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On learning mathematics[1]

JEROME S. BRUNER, *Harvard University, Cambridge, Massachusetts.*

What places do discovery, intuition, and translation have in the learning of mathematics? Are our present activity and efforts to improve mathematics teaching the beginning of an educational renaissance?

I AM CHALLENGED and honored to be asked to speak before a group of teachers of mathematics on the nature of learning and thinking—particularly mathematical learning and thinking. Let me introduce you to my intentions by citing a remark of the English philosopher, Weldon. He noted that one could discriminate between difficulties, puzzles, and problems. A difficulty is a trouble with minimum definition. It is a state in which we know that we want to get from here to there, both points defined rather rawly, and with not much of an idea how to bridge the gap. A puzzle, on the other hand, is a game in which there is a set of givens and a set of procedural constraints, all precisely stated. A puzzle also requires that we get from here to there, and there is at least one admissible route by which we can do so, but the choice of route is governed by definite rules that must not be violated. A typical puzzle is that of the Three Cannibals and Three Missionaries, in which you must get three missionaries and three cannibals across a river in a boat that carries no more than two passengers. You can never have more cannibals than missionaries on one side at a time. Only one cannibal can row; all three missionaries can. Another puzzle, one in which the terminus has not yet been achieved, is the so-called Twin Primes Conjecture. Now, Weldon proposes that a problem is a difficulty upon which we attempt to impose a puzzle form. A young man, trying to win the favor of a young lady—a difficulty—decides to try out successively and with benefit of correction by experience, a strategy of flattery—an iterative procedure, and a classic puzzle—and thus converts his difficulty into a problem. I rather expect that most young men do all this deciding at the unconscious level; I hope so for the sake of my daughters! But the point of mentioning it is not my fatherly jealousy, but to emphasize that the conversion of difficulties into problems by the imposition of puzzle forms is often not always done with cool awareness, and that part of the task of the mathematician is to work toward an increase in such awareness. But this gets me ahead of my exposition.

Let me urge that the pure mathematician is above all a close student of puzzle forms—puzzles involving the ordering of sets of elements in a manner to fulfill specifications. The puzzles, once grasped, are obvious, so obvious that it is astounding that anybody has difficulty with mathematics at all, as Bertrand Russell once said in exasperation. "Why, the rowing cannibal takes over another cannibal and returns. Then he takes over the other cannibal and returns. Then two missionaries go over, and one of them brings back a nonrowing cannibal. Then a missionary takes the rowing cannibal over and brings back a nonrowing cannibal. Then two missionaries go over and stay,

[1] A paper presented before The National Council of Teachers of Mathematics, Salt Lake City, Utah, August 1960.

while the rowing cannibal travels back and forth, bringing the remaining cannibals over one at a time. And there are never more cannibals than missionaries on either side of the river." It is simple. If you say that my statement of the solution is clumsy and lacking in generality, even though correct, you are quite right. But now we are talking mathematics.

For the mathematician's job is not pure puzzle-mongering. It is to find the deepest properties of puzzles so that he may recognize that a particular puzzle is an exemplar—trivial, degenerate, or important, as the case may be—of a family of puzzles. He is also a student of the kinship that exists between families of puzzles. So, for example, he sets forth such structural ideas as the commutative, associative, and distributive laws to show the manner in which a whole set of seemingly diverse problems all have a common puzzle form imposed on them.

It is probably the case that there are two ways in which one goes about both learning mathematics and teaching it. One of them is through a technique that I want to call unmasking: discovering the abstracted ordering properties that lie behind certain empirical problem solutions in the manner in which the triangulation techniques used for reconstructing land boundaries in the Nile valley eventually developed into the abstractions of plane geometry, having first been more like surveying than mathematics. Applied mathematics, I would think, is still somewhat similar in spirit, although I do not wish to become embroiled in the prideful conflict over the distinction between pure and applied. The more usual way in which one learns and teaches is to work directly on the nature of puzzles themselves—on mathematics *per se*.

I should like to devote my discussion to four topics related to the teaching or learning of mathematics. The first has to do with the role of *discovery*, wherein it is important or not that the learner discover things for himself. I have been both puzzled (or I should say "difficultied") and intrigued hearing some of you discuss this interesting matter. The second topic is *intuition*, the class of nonrigorous ways by which mathematicians speed toward solutions or cul-de-sacs. The third is mathematics as an analytic language, and I should like to concentrate on the problem of the *translation* of intuitive ideas into mathematics. I hope you will permit me to assume that anything that can be said in mathematical form can also be said in ordinary language, though it may take a tediously long time to say it and there will always be the danger of imprecision of expression. The fourth and final problem is the matter of *readiness:* when is a child "ready" for geometry or topology or a discussion of truth tables? I shall try to argue that readiness is factorable into several more familiar issues.

Discovery

I think it can be said now, after a decade of experimentation, that any average teacher of mathematics can do much to aid his or her pupils to the discovery of mathematical ideas for themselves. Probably we do violence to the subtlety of such technique by labelling it simply the "method of discovery," for it is certainly more than one method, and each teacher has his own tricks and approach to stimulating discovery by the student. These may include the use of a Socratic method, the devising of particularly apt computation problems that permit a student to find regularities, the act of stimulating the student to short cuts by which he discovers for himself certain interesting algorisms, even the projection of an attitude of interest, daring, and excitement. Indeed, I am struck by the fact that certain ideas in teaching mathematics that take a student away from the banal manipulation of natural numbers have the effect of freshing his eye to the possibility of discovery. I interpret such trends as the use of set theory in the early grades partly in this light—so too the Cuisenaire rods, the

use of modular arithmetic, and other comparable devices.

I know it is difficult to say when a child has discovered something for himself. How big a leap must he take before we will grant that a discovery has been made? Perhaps it is a vain pursuit to try to define a discovery in terms of what has been discovered by whom. Which is more of a discovery—that $3+4=7$, that $3x+4x=7x$, or that 7 shares with certain other sets the feature that it cannot be arranged in rectangular ranks? Let me propose instead that discovery is better defined not as a product discovered but as a process of working, and that the so-called method of discovery has as its principal virtue the encouragement of such a process of working or, if I may use the term, such an attitude. I must digress for a moment to describe what I mean by an attitude of discovery, and then I shall return to the question of why such an attitude may be desirable not only in mathematics but as an approach to learning generally.

In studying problem solving in children between the ages of 11 and 14, we have been struck by two approaches that are almost polar opposites. Partly as an analogy, but only partly, we have likened them, respectively, to the approach of a listener and the approach of a speaker toward language. There are several interesting differences between the two. The listener's approach is to take the information he receives in the order in which it comes; he is bound in the context of the flow of speech he is receiving, and his effort is to discern a pattern in what comes to him. Perforce, he lags a bit behind the front edge of the message, trying to put the elements of a moment ago together with those that are coming up right now. The listener is forced into a somewhat passive position since he does not have control of the direction of the message or of its terminus. It is interesting that listeners sometimes fall asleep. It is rare for a speaker to fall asleep. For the speaker is far more active. He, rather than lagging behind the front edge of the message he is emitting, is well out ahead of it so that the words he is speaking lag behind his thoughts. He decides upon sequence and organization.

Now a wise expositor knows that to be effective in holding his auditor he must share some of his role with him, must give him a part in the construction game by avoiding monologue and adopting an interrogative mode when possible. If he does not, the listener either becomes bored or goes off on his own internal speaking tour.

Some children approach problems as a listener, expecting to find an answer or at least some message there. At their best they are receptive, intelligent, orderly, and notably empirical in approach. Others approach problem solving as a speaker. They wish to determine the order of information received and the terminus of their activity and to march ahead of the events they are observing. It is not only children. As a friend of mine put it, a very perceptive psychologist indeed, some men are more interested in their own ideas, others are more interested in nature. The fortunate ones care about the fit between the two. Piaget, for example, speaks of the two processes of accommodation and assimilation, the former being a process of accepting what is presented and changing with it, the latter being the act of converting what one encounters into the already existing categories of one's thought. Each attitude has its excesses. The approach of the listener can become passive and without direction. The approach of the speaker can become assimilative to the point of autistic thinking. As Piaget points out in his brilliant studies of thinking in early childhood, some sort of balance between the two is essential for effective cognitive functioning.

It is in the interest of maintaining this balance that, I would propose, the approach of discovery is centrally important. The overly passive approach to learning, the attitude of the listener, creates a

situation in which the person expects order to come from outside, to be in the material that is presented. Mathematical manipulation requires reordering, unmasking, simplification, and other activities akin to the activity of a speaker.

There is one other thing that I would emphasize about discovery: its relation to reward and punishment. I have observed a fair amount of teaching in the classroom: not much, but enough to know that a great deal of the daily activity of the student is not rewarding in its own right. He has few opportunities to carry a cycle of working or thinking to a conclusion, so that he may feel a sense of mastery or of a job well done. At least when he makes a paper airplane, he can complete the cycle almost immediately and know whether or not the thing flies. It is not surprising then that it is necessary to introduce a series of extrinsic rewards and punishments into school activity—competition, gold stars, etc.—and that, in spite of these, there are still problems of discipline and inattention. Discovery, with the understanding and mastery it implies, becomes its own reward, a reward that is intrinsic to the activity of working. I have observed and even taught classes in which the object was to stimulate discovery, and I have seen masterful teachers accomplish it. I am impressed by the fact that, although competitive advantage is still strong in such a classroom atmosphere, it is nonetheless the case that the experience of discovering something, even if it be a simple short cut in computation, puts reward into the child's own hands.

I need not tell you that there are practical difficulties. One cannot wait forever for discovery. One cannot leave the curriculum entirely open and let discovery flourish willy-nilly wherever it may occur. What kinds of discoveries to encourage? Some students are troubled and left out and have a sense of failure. These are important questions, but they should be treated as technical and not as substantive ones. If emphasis upon discovery has the effect of producing a more active approach to learning and thinking, the technical problems are worth the trouble.

Intuition

It is particularly when I see a child going through the mechanical process of manipulating numbers without any intuitive sense of what it is all about that I recall the lines of Lewis Carroll: "Reeling and Writhing, of course, to begin with . . . and then the different branches of Arithmetic—Ambition, Distraction, Uglification, and Derision." Or as Max Beberman puts it, much more gently, "Somewhat related to the notion of discovery in teaching is our insistence that the student become aware of a concept before a name has been assigned to the concept."[2] I am quite aware that the issue of intuitive understanding is a very live one among teachers of mathematics and even a casual reading of the Twenty-fourth Yearbook[3] of your Council makes it clear that you are also very mindful of the gap that exists between proclaiming the importance of such understanding and actually producing it in the classroom.

Intuition implies the act of grasping the meaning or significance or structure of a problem without explicit reliance on the analytic apparatus of one's craft. It is the intuitive mode that yields hypotheses quickly, that produces interesting combinations of ideas before their worth is known. It precedes proof; indeed, it is what the techniques of analysis and proof are designed to test and check. It is founded on a kind of combinatorial playfulness that is only possible when the consequences of error are not overpowering or sinful. Above all, it is a form of activity that depends upon confidence in the worthwhileness of the process of mathe-

[2] Max Beberman, *An Emerging Program of Secondary School Mathematics* (Cambridge, Massachusetts: Harvard University Press, 1958), p. 33.

[3] *The Growth of Mathematical Ideas, Grades K-12*, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: The National Council, 1959).

matical activity rather than upon the importance of right answers at all times.

I should like to examine briefly what intuition might be from a psychological point of view and to consider what we can possibly do about stimulating it among our students. Perhaps the first thing that can be said about intuition when applied to mathematics is that it involves the embodiment or concretization of an idea, not yet stated, in the form of some sort of operation or example. I watched a ten-year-old playing with snail shells he had gathered, putting them into rectangular arrays. He discovered that there were certain quantities that could not be put into such a rectangular compass, that however arranged there was always "one left out." This of course intrigued him. He also found that two such odd-man-out arrays put together produced an array that was rectangular, that "the left out ones could make a new corner." I am not sure it is fair to say this child was learning a lot about prime numbers. But he most certainly was gaining the intuitive sense that would make it possible for him later to grasp what a prime number is and, indeed, what is the structure of a multiplication table.

I am inclined to think of mental development as involving the construction of a model of the world in the child's head, an internalized set of structures for representing the world around us. These structures are organized in terms of perfectly definite grammars or rules of their own, and in the course of development the structures change and the grammar that governs them also changes in certain systematic ways. The way in which we gain lead time for anticipating what will happen next and what to do about it is to spin our internal models just a bit faster than the world goes.

Now the child whose behavior I was just describing had a model of quantities and order that was implicitly governed by all sorts of seemingly subtle mathematical principles, many of them newly acquired and some of them rather strikingly original. He may not have been able to talk about them, but he was able to do all sorts of things on the basis of them. For example, he had "mastered" the very interesting idea of conservation of quantity across transformations in arrangement or, as you would say, the associative law. Thus, the quantity 6 can be stated as $2+2+2$, $3+3$, and by various "irregular" arrangements, as $2+4$, $4+2$, $2+(3+1)$, $(2+3)+1$, etc. Inherent in what he was doing was the concept of reversibility, as Piaget calls it, the idea of an operation and its inverse. The child was able to put two sets together and to take them apart; by putting together two prime number arrays, he discovers that they are no longer prime (using our terms now) but can be made so again by separation. He was also capable of mapping one set uniquely on another, as in the construction of two identical sets, etc. This is a formidable amount of highbrow mathematics.

Now what do we do with this rather bright child when he gets to school? Well, in our own way we communicate to him that mathematics is a logical discipline and that it has certain rules, and we often proceed to teach him algorisms that make it seem that what he is doing in arithmetic has no bearing on the way in which one would proceed by nonrigorous means. I am not, mind you, objecting to "social arithmetic" with its interest rates and baseball averages. I am objecting to something far worse, the premature use of the language of mathematics, its end-product formalism, that makes it seem that mathematics is something new rather than something the child already knows. It is forcing the child into the inverse plight of the character in *Le Bourgeois Gentilhomme* who comes to the blazing insight that he has been speaking prose all his life. By interposing formalism, we prevent the child from realizing that he has been thinking mathematics all along. What we do, in essence, is to remove his confidence in his ability to perform the processes of

mathematics. At our worst, we offer formal proof (which is necessary for checking) in place of direct intuition. It is good that a student know how to check the conjecture that $8x$ is equivalent to the expression $3x+5x$ by such a rigorous statement as the following: "By the commutative principle for multiplication, for every x, $3x+5x=x3+x5$. By the distributive principle, for every x, $x3+x5=x(3+5)$. Again by the commutative principle, for every x, $x(3+5)=(3+5)x$ or $8x$. So, for every x, $3x+5x=8x$." But it is hopeless if the student gets the idea that this and this only is *really* arithmetic or algebra or "math" and that other ways of proceeding are really for nonmathematical slobs. Therefore, "mathematics is not for me."

I would suggest, then, that it is important to allow the child to use his natural and intuitive ways of thinking, indeed to encourage him to do so, and to honor him when he does well. I cannot believe that he has to be taught to do so. Rather, we would do well to end our habit of inhibiting the expression of intuitive thinking and then to provide means for helping the child to improve in it. To this subject I turn next.

Translation

David Page wrote me last year: "When I tell mathematicians that fourth grade students can go a long way into 'set theory,' a few of them reply, 'Of course.' Most of them are startled. The latter ones are completely wrong in assuming that set theory is intrinsically difficult. Of course, it may be that nothing is intrinsically difficult—we just have to wait the centuries until the proper point of view and corresponding language is revealed!" How can we state things in such a way that ideas can be understood and converted into mathematical expression?

It seems to me there are three problems here. Let me label them the *problem of structure*, the *problem of sequence*, and the *problem of embodiment*. When we try to get a child to understand a concept, leaving aside now the question of whether he can "say" it, the first and most important problem, obviously, is that we as expositors understand it ourselves. I apologize for making such a banal point, but I must do so, for I think that its implications are not well understood. To understand something well is to sense wherein it is simple, wherein it is an instance of a simpler, general case. I know that there are instances in the development of knowledge in which this may not prove to be the case, as in physics before Mendeleev's table or in contemporary physics where particle theory is for the moment seemingly moving toward divergence rather than convergence of principles. In the main, however, to understand something is to sense the simpler structure that underlies a range of instances, and this is notably true in mathematics.

In seeking to transmit our understanding of such structure to another person—be he a student or someone else—there is the problem of finding the language and ideas that the other person would be able to use if he were attempting to explain the same thing. If we are lucky, it may turn out that the language we would use would be within the grasp of the person we are teaching. This is not, alas, always the case. We may then be faced with the problem of finding a homologue that will contain our own idea moderately well and get it across to the auditor without too much loss of precision, or at least in a form that will permit us to communicate further at a later time.

Let me provide an example. We wish to get across to the first-grade student that much of what we speak of as knowledge in science is indirect, that we talk about such things as pressure or chemical bonds or neural inhibition although we never encounter them directly. They are inferences we draw from certain regularities in our observations. This is all very familiar to us. It is an idea with a simple structure but with complicated implications. To a young student who is used to thinking of

things that either exist or do not exist, it is hard to tell the truth in answer to his question of whether pressure "really" exists. We wish to transmit the idea that there are observables that have regularities and constructs that are used for conserving and representing these regularities, that both, in different senses, "exist," and the constructs are not fantasies like gremlins or fairies. That is the structure.

Now there is a sequence. How do we get the child to progress from his present two-value logic of things that exist and things that do not exist to a more subtle grasp of the matter? Take an example from the work of Inhelder and Piaget. They find that there are necessary sequences or steps in the mastery of a concept. In order for a child to understand the idea of serial ordering, he must first have a firm grasp on the idea of comparison—that one thing includes another or is larger than another. Or, in order for a child to grasp the idea that the angle of incidence is equal to the angle of reflection, he must first grasp the idea that for any angle at which a ball approaches a wall, there is a corresponding unique angle by which it departs. Until he grasps this idea, there is no point in talking about the two angles being equal or bearing any particular relationship to each other, just as it is a waste to try to explain transitivity to a child who does not yet have a firm grasp on serial ordering.

The problem of embodiment then arises: how to embody illustratively the middle possibility of something that does not quite exist as a clear and observable datum? Well, one group of chemists working on a new curriculum proposed as a transitional step in the sequence that the child be given a taped box containing an unidentified object. He may do anything he likes to the box: shake it, run wires through it, boil it, anything but open it. What does he make of it? I have no idea whether this gadget will indeed get the child to the point where he can then more easily make the distinction between constructs and data. But the attempt is illustrative and interesting. It is a nice illustration of how one seeks to translate a concept (in this instance the chemical bond) into a simpler homologue, an invisible object whose existence depended upon indirect information, by the use of an embodiment. From there one can go on.

The discussion leads me immediately to two practical points about teaching and curriculum design. The first has to do with the sequence of a curriculum, the second with gadgetry. I noted with pleasure in the introductory essay of the Twenty-fourth Yearbook of the National Council of Teachers of Mathematics that great emphasis was placed upon continuity of understanding: "Theorem 2. Teachers in all grades should view their task in the light of the idea that the understanding of mathematics is a continuum. . . . This theorem implies immediately the corollaries that: (1) Teachers should find what ideas have been presented earlier and deliberately use them as much as possible for the teaching of new ideas. (2) Teachers should look to the future and teach some concepts and understandings even if complete mastery cannot be expected."[3] Alas, it has been a rarity to find such a structure in the curriculum, although the situation is likely to be remedied in a much shorter time than might have been expected through the work of such organizations as the School Mathematics Study Group. More frequently fragments are found here and there: a brilliant idea about teaching co-ordinate systems and graphing, or what not. I have had occasion to look at the list of teaching projects submitted to the National Science Foundation. There is everything from a demonstrational wind tunnel to little Van de Graaff generators, virtually all divorced from any sequence. Our impulse is toward gadgetry. The need instead is for something approximating a spiral curriculum, in which ideas are presented in homologue form, returned to later with more preci-

sion and power, and further developed and expanded until, in the end, the student has a sense of mastery over at least some body of knowledge.

There is one part of the picture in the building of mathematical curriculum now in progress where I see a virtual blank. It has to do with the investigation of the language and concepts that children of various ages use in attempting intuitively to grasp different concepts and sequences in mathematics. This is the language into which mathematics will have to be translated while the child is en route to more precise mastery. The psychologist can help in all this, it seems to me, as a handmaiden to the curriculum builder, by devising ways of bridging the gap between ideas in mathematics and the students' ways of understanding such ideas. His rewards will be rich, for he not only will be helping education toward greater effectiveness, but also will be learning afresh about learning. If I have said little to you today about the formal psychology of learning as it now exists in many of our university centers, it is because most of what exists has little bearing on the complex and ordered learning that you deal with in your teaching.

Readiness

One of the conclusions of the Woods Hole Conference of the National Academy of Sciences on curriculum in science was that any subject can be taught to anybody at any age in some form that is honest.[4] It is a brave assertion, and the evidence on the whole is all on its side. At least there is no evidence to contradict it. I hope that what I have had to say about intuition and translation is also in support of the proposition.

Readiness, I would argue, is a function not so much of maturation—which is not to say that maturation is not important—but rather of our intentions and our skill at translation of ideas into the language and concepts of the age we are teaching. But let it be clear to us that our intentions must be plain before we can start deciding what can be taught to children of what age, for life is short and art is long and there is much art yet to be created in the transmission of knowledge. So let me say a word about our intentions as educators.

When one sits down to the task of trying to write a textbook or to prepare a lesson plan, it soon becomes apparent—at whatever level one is teaching—that there is an antinomy between two ideals: coverage and depth. Perhaps this is less of a problem in mathematics than in the field of history or literature, but not by any means is it negligible. In content, positive knowledge is increasing at a rate that, from the point of view of what portion of it one man can know in his lifetime, is, to some, alarming. But at the same time that knowledge increases in its amount, the degree to which it is structured also increases. In Robert Oppenheimer's picturesque phrase, it appears that we live in a "multi-bonded pluriverse" in which, if everything is not related to everything else, at least everything is related to something. The only possible way in which individual knowledge can keep proportional pace with the surge of available knowledge is through a grasp of the relatedness of knowledge. We may well ask of any item of information that is taught or that we lead a child to discover for himself whether it is worth knowing. I can only think of two good criteria and one middling one for deciding such an issue: whether the knowledge gives a sense of delight and whether it bestows the gift of intellectual travel beyond the information given, in the sense of containing within it the basis of generalization. The middling criterion is whether the knowledge is useful. It turns out, on the whole, as Charles Sanders Peirce commented, that useful knowledge looks after itself. So I would urge that we as school men let it do so and concentrate on the first

[4] Jerome S. Bruner, *The Process of Education* (Cambridge, Massachusetts: Harvard University Press, 1960).

two criteria. Delight and travel, then.

It seems to me that the implications of this conclusion are that we opt for depth and continuity in our teaching rather than coverage, and that we re-examine afresh what it is that bestows a sense of intellectual delight upon a person who is learning. To do the first of these, we must ask what it is that we wish the man in our times to know, what sort of minimum. What do we mean by an educated man? There is obviously not time now to examine this question in the detail it deserves. But I think we would all agree that, at the very least, an educated man should have a sense of what knowledge is like in some field of inquiry, to know it in its connectedness and with a feeling for how the knowledge is gained. An educated man must not be dazzled by the myth that advanced knowledge is the result of wizardry. The way to battle this myth is in the direct experience of the learner—to give him the experience of going from a primitive and weak grasp of some subject to a stage in which he has a more refined and powerful grasp of it. I do not mean that each man should be carried to the frontiers of knowledge, but I do mean that it is possible to take him far enough so that he himself can see how far he has come and by what means.

If I may take a simple example, let me use the principles of conservation in physics: the conservation of energy, mass, and momentum. Indeed, I would add to the list the idea of invariance across transformation in order to include mathematics more directly. The child is told, by virtue of living in our particular society and speaking our particular language, that he must not waste his energy, fritter it away. In common experience, things disappear, get lost. Bodies "lose" their heat; objects set in motion do not appear to stay in motion as in the pure case of Newton's law. Yet, the most powerful laws of physics and chemistry are based on the conception of conservation. Only the meanest of purists would argue against the effort to teach the conservation principles to a first-grade student on the grounds that it would be "distorted" in the transmission. We know from the work of Piaget and others that, indeed, the child does not easily agree with notions based on conservation. A six-year-old child will often doubt that there is the same amount of fluid in a tall, thin glass jar as there was in a flat, wide one, even though he has seen the fluid poured from the latter into the former. Yet, with time and with the proper embodiment of the idea—as in the film of the Physical Science Study Committee where a power plant is used as an example—the idea can be presented in its simplest and weakest form.

Let the idea be revisited constantly. It is central to the structure of the sciences of nature. In good time, many things can be derived from it that yield tremendous predictive power. Coverage in this sense, that is, showing the range of things that can be related to this particular and powerful something, serves the ends of depth. But what of delight? If you should ask me as a student of the thought processes what produces the most fundamental form of pleasure in man's intellectual life, I think I would reply that it is the reduction of surprise and complexity to predictability and simplicity. Indeed, it is when a person has confidence in his ability to bring off this feat that he comes to enjoy surprise, to enjoy the process of imposing puzzle forms upon difficulties in order to convert them into problems. I think we as educators recognized this idea in our doctrine of the "central subject," the idea of co-ordinating a year's work around a central theme. But choosing a central theme horizontally, for the year's work, is arbitrary and often artificial. The central themes are longitudinal. The most important central theme is growth in your own sense of mastery, of knowing today that you have more power and control and mastery over a subject than you had last year. If we produce such a sense of growth, I think it produces de-

light in knowledge as a by-product automatically.

My choice of the conservation theorems as an illustration was not adventitious. I tried to choose one as basic to the natural sciences as one could make it. Similar themes recur and have eventual crescendo value in other fields: the idea of biological continuity whereby giraffes have giraffe babies and not elephant babies, the idea of tragedy in literature, the notion of the unit of measure in mathematics, the idea of chance as a fraction of certainty in statistics, the grammar of truth tables in logic. It would seem to be altogether appropriate to bring about a joining of forces of experienced teachers, our most gifted scholars, and psychologists to see what can be done to structure longitudinal curricula of this order.

When we are clear about what we want to do in this kind of teaching, I feel reasonably sure that we will be able to make rapid strides ahead in dealing with the pseudoproblem of readiness. I urge that we use the unfolding of readiness to our advantage: to give the child a sense of his own growth and his own capacity to leap ahead in mastery. The problem of translating concepts to this or that age level can be solved, the evidence shows, once we decide what it is we want to translate.

I have perhaps sounded optimistic in my remarks. The evidence warrants optimism, and I cannot help but feel that we are on the threshold of a renaissance in education in America. Let me recapitulate my argument briefly. With the active attitude that an emphasis on discovery can stimulate, with greater emphasis (or fewer restraints) on intuition in our students, and with a courteous and ingenious effort to translate organizing ideas into the available thought forms of our students, we are in a position to construct curricula that have continuity and depth and that carry their own reward in giving a sense of increasing mastery over powerful ideas and concepts that are worth knowing, not because they are interesting in a trivial sense but because they give the ultimate delight of making the world more predictable and less complex. It is this perspective that makes me optimistic and leads me to believe that our present flurry is the beginning not of another fad, but of an educational renaissance.

Conference Board sponsors "Continental Classroom" course

The Conference Board of the Mathematical Sciences is the mathematics sponsor of Contemporary Mathematics, the new "Continental Classroom" course for 1960–61. The other sponsors are Learning Resources Institute and the National Broadcasting Company. The course began on September 26 and runs for 32 weeks, and it is being presented from 6:30 to 7:00 A.M., Monday through Friday, in each time zone. The first semester of Contemporary Mathematics will be devoted to Modern Algebra, by Professor John L. Kelley with the assistance of Dr. Julius Hlavaty; the second semester of the course will be devoted to Probability and Statistics, by Professor Frederick Mosteller with the assistance of Professor Paul C. Clifford.

The Conference Board appointed the following advisory committee, which has assisted Learning Resources Institute in planning Contemporary Mathematics: E. G. Begle (Chairman), L. W. Cohen, R. P. Dilworth, P. S. Jones, J. R. Mayor, E. J. McShane, A. E. Meder, Jr., and Mina S. Rees.

Some geometric ideas for junior high school

IRVIN H. BRUNE, *Iowa State Teachers College, Cedar Falls, Iowa.*

"The race, including those who advocate abstractions today, learned much of its mathematics through practical needs and by the way of real problems. . . . Surely a small share of such learning while solving, abstracting while applying, and generalizing while testing should not be denied young people today."

Variety via geometry

At times pupils need a new start. This applies particularly to mathematics in the junior high school. There, most pupils are ready for something different—a change from mathematics as computing, however important such exercises may be.

Geometry provides a fresh look. Especially in the junior high school it is the study of form. It involves shapes, forms, and patterns. It enables pupils to realize that mathematics is more than the manipulating of marks. When taught with an eye to its helping pupils to construct graphs, geometry supports the study of sets, sentences, and statements.

The present discussion mentions several activities and illustrates a few in detail. The stress falls on geometry as form.

Growth in geometry

Awareness of shapes and sizes arises at an early age. A board with cutouts and a set of blocks to match the cutouts gets a lively reception from most kindergartners; they plop the figures into the proper places with dispatch. Already these children are shape conscious.

As they continue in the elementary grades, children enlarge their understandings of circles, triangles, cones, rectangles, cylinders, spheres, cubes, squares, and so on.

All of this is good. It may even refute the claim, made by many, that elementary mathematics overplays drill, mechanics, and memory work. At least as pupils handle solids—real materials—and sketch figures in one, two, and three dimensions, they learn about forms via their own discoveries.

Reality in geometry

Indeed, in pupils' discoveries about real things lies the heart of the matter. Why is this so? Why stress reality in geometry? For one thing, with our present commendable emphasis on acceleration—preparing pupils, for example, to study calculus successfully when they enter college—we can get too grim. Geometry for adults properly enough consists of deductions drawn from a set of postulates pertaining to a set of undefined elements. It emphasizs composition rather than content. Whether geometers deal with points and lines, or with punks and gerds, the structure persists.

But thereby we take risks. In our zeal to teach pure mathematics, we neglect applied mathematics. And, in our eagerness to move pupils along, we foist all manner of abstractions on them. What matters more, however, is the pupils' appreciation of geometry as the science of space, including the world in which they

live and other worlds to which they may hope to travel.

Indeed, these pupils need a concrete approach: problems about land, sea, and air, and experiments with triangles, quadrilaterals, circles, etc. Despite teachers' present laudable zeal to hasten their pupils' mathematical maturity, they need not discard experimentation and discovery. Intuitive geometry still has a place. Pupils in junior high school still have a right to learn that geometry deals first of all with practical affairs. It properly begins with real objects, with models pupils can build, with pictures they can draw.

History and geometry

Geometry, one should not forget, began with earth measuring. Dwellers on the river banks of antiquity built levees, dams, and reservoirs, and dug ditches and canals.

Prehistoric minds needed to develop a geometry of land measuring, planet observing, and monument building. Not until millennia later, with the curious and studious Greeks, did abstract geometry inch into men's consciousness. Even then, only the learned few had the time and the desire to develop theoretical mathematics.

Eventually the Greeks did create and transmit geometry as deduction. This work has endured. It also frequently evokes the cry that the curriculum, which cleaves to Euclid's apparently deathless geometry, needs modernization. Indeed, the structures and the way of thinking known today as modern mathematics came into being mostly during the past century.

So what shall it be today? The point in the present discussion is that children, like their ancient forbears, first need informal geometry. Whatever modernization comes about should not deny pupils the learning they can forge for themselves with geometric figures.

Proof in geometry

This does not mean that the concept of proof dies from neglect. Indeed, pupils should steadily develop ideas of proof at appropriate levels as they progress from kindergarten through senior high school—from counting to measuring to deducing. Modern materials, new experiences, and up-to-date ideas should contribute to the growth of this major concept. But pushing for mathematical maturity should not result in pupils' skipping informal experiments with cubes, spheres, cylinders, and cones. They should not miss the satisfactions they can gain from measuring circles, triangles, quadrilaterals, lines, and angles. It is a kind of reliving of a little of the rule-of-thumb mathematics the race painfully produced. Thus by conducting pupils through a tiny fraction of the steps man has taken to develop geometry as the science of space, teachers can encourage pupils to experiment, report, and generalize before they devise formal proofs. In such work, motivation for subsequent more rigorous proof of the relations flourishes.

Everyday geometry

Hollow cones, cubes, cylinders, and boxes of various shapes can be made by the pupils themselves. So can the basic unit of volume, the cubic inch. At times a father-son woodworking team can supply cubic inches in quantity. Inch cubes are also available commercially. So, too, are models of various solids, including the sphere, which is not exactly easy for pupils to construct. Comparisons of volumes of various hollow solids via sand pouring are not a new story. The principle, of course, is that pupils learn from handling, drawing, and building. They discover important relations and report their findings to their classmates.

Similar considerations apply to the geometry of the plane. What does it profit a pupil repeatedly to figure areas and perimeters if he lacks the concept, square inch? Even though he used inch squares as counters in the primary grades, he ought to make and handle at least a few such units before he evaluates formulas.

Better still, he should figure those formulas out for himself. We turn now to some other activities.*

Triangles

Pupils can begin with one, two, and three line segments. Under what conditions do plane figures appear? Apparently the triangle is the simplest polygon, and,

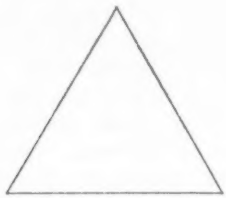


Figure 1

among triangles, the equilateral is the simplest (Fig. 1).

What happens if one shoves the apex of such a triangle up and down? Figure 2 shows the results of doubling and redoubling the altitude. It also shows the effect of choosing altitudes that are binary submultiples of the original altitude. In this exercise pupils readily see good reason for learning how to bisect a segment. They also meet isosceles triangles in some abundance here. Moreover, reflections of these triangles, reflections beneath the fixed base of the original triangle, provide further exploring. The line of apices then provides an introduction to directed segments; it illustrates positive and negative segments having absolute values of 1, 2, 4, 8, etc.

Shoving the apex can also proceed toward the right (and left) along a line parallel to the base of the triangle. Figure 3 shows this. If displacement distances of $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, and so on, times the length of a

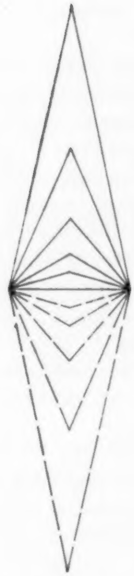


Figure 2

side of the original triangle are used, the pupil meets right, isosceles, obtuse, and scalene triangles in the process. He also tends to see why people may want a foolproof way accurately to construct a line parallel to a given line.

Moreover, the question might well arise, "When will the ever-skinnier triangles merge into single straight lines?" The notion of ideal points in the plane may

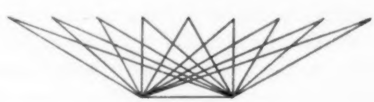


Figure 3

well appear here and, by dint of the pupil's uncovering it himself, become clear.

Besides experiences with triangles, parallelograms and rhombuses appear. Areas and diagonals also come in. That diagonals of parallelograms bisect each other (perpendicularly in the case of the rhombus) may also come to the notice of the pupils. See Figures 4–6.

A lesson in language might interest the

* For a much fuller treatment of many of the ideas that follow and for a wealth of further examples see H. von Baravalle, *Geometrie als Sprache der Formen* (Freiburg im Breisgau, Germany: Novalis Verlag, 1957). An English translation is now being prepared and will soon be available from Saint George's Guild, Paterson, New Jersey. Dr. Baravalle has graciously given permission for the use of his ideas in this paper.

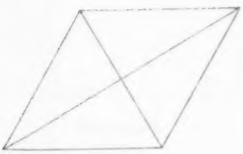


Figure 4

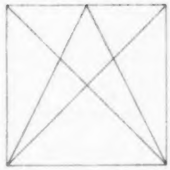


Figure 5

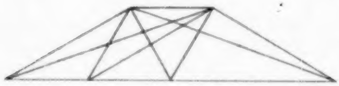


Figure 6

pupils here (Fig. 4). Since the plural of "abacus" is "abaci" more often than it is "abacuses," how about the plural of "rhombus"? Some minor research will probably disclose that "rhombuses" seems slightly to outdistance "rhombi" in common usage. Both, however, are correct.

QUADRILATERALS

Speaking of rhombuses suggests that other quadrilaterals provide pupils with opportunities to learn things. Just as the equilateral triangle provides the utmost in triangular regularity, so also the square outdoes the other quadrilaterals. Equipped with ruler, compasses, and protractor and confronted with a square they have just drawn, pupils can discover, besides equal sides, equal angles, and equal diagonals, the further equal angles and triangles that the diagonals form (Fig. 5).

The simple question, "Is a square a rectangle?" sometimes evokes discussion

Figures 7–9

that is terminable, perhaps, only by reference to an authority. Here, too, the full import of "rect-angle" may appear.

Altering the square but slightly produces either a more general rectangle or a rhombus (Figs. 7–9). Changing from a square by altering lengths of sides *and* sizes of angles results in a more general parallelogram (Fig. 10). Pupils note in these cases that such changes proceed from four equal sides and four equal angles to two pairs of equal sides and/or equal angles. To mention language again, rarely do pupils in secondary school mathematics know that the rhombus has two other names, "diamond" and "lozenge."

Besides having sides and angles equal in pairs, parallelograms derive their very name from the property of parallel sides. If the equality of two pairs of angles and one pair of sides be maintained, together with exactly one pair of parallel sides, the

Figures 10–12

figure becomes an isosceles trapezoid (Fig. 11). If two angles and two pairs of sides remain equal, but the sides are not parallel, the parallelogram converts into a kite-shaped figure (Fig. 12). In Figure 13

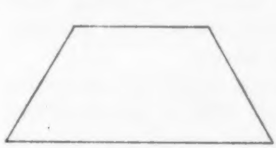


Figure 13

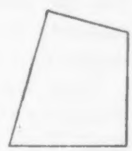


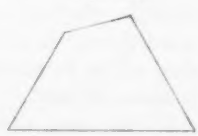
Figure 14

exactly three sides of the quadrilateral are equal.

The quadrilateral may contain exactly two right angles, and no two sides either parallel or equal (Fig. 14). Or the figure may possess two parallel sides and no two sides or angles equal (Fig. 15). Or the

Figure 15

Figure 16



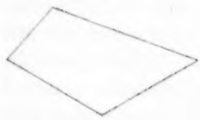


Figure 17

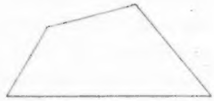


Figure 18

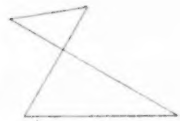


Figure 19

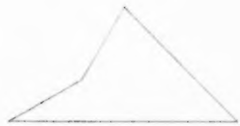


Figure 20

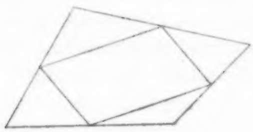


Figure 21

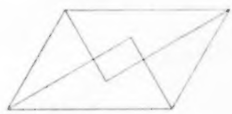


Figure 22

Figure 23

situation may involve exactly two equal angles and no sides equal or parallel (Fig. 16). Or exactly two sides may be equal, with none parallel and no two angles equal (Fig. 17).

A quadrilateral, furthermore, may be entirely irregular, having no parallelism and no equality of sides or angles (Fig. 18). Unless the definition of quadrilateral, moreover, excludes intersections of sides at a point other than a vertex (Fig. 19) as well as interior angles larger than a straight angle, the figure may be either concave (Fig. 20) or re-entrant (Fig. 19). For the latter case, the German *überschlagen*, meaning *crossed*, or, literally, *thrown over*, fits neatly.

What happens if pupils join successive midpoints of the sides of any quadrilateral? What if they extend the bisectors of the angles of a general parallelogram? Of a general rectangle? Can one bisector meet another? Can it meet two others? Three others?

Figures 21, 22, and 23 show some answers. Here again the concept of square versus rectangle merits attention. Is the interior figure a rectangle? Is it a square? If pupils keep track of the sizes of angles, and if they have already accepted 180° as the sum of the measures of the angles in a triangle, they can *prove* that the inside quadrilateral is at least a rectangle. Can they go on to satisfy their hunch that it is also a square?

If they can, then they will have progressed, eventually, from square to rectangle to parallelogram to irregular quadrilateral and then, in reverse, from irregular quadrilateral back to square again.

The coins

Suppose a pupil investigates the question, "How many dimes will you need to form a ring of dimes around a dime, all just touching?" (Fig. 24). How about nickels, quarters, cents?

As Figure 25 suggests, this discovery bears on the construction of a regular hexagon, which in turn involves equilateral triangles, six-pointed stars, and six-leaved roses (Figs. 26–28). Also, of course, hexagons and dodecagons bear intimate relations.

Although pupils can readily trisect a given circle by joining alternate vertices of an inscribed hexagon, some experimenting will probably lead them to a short cut. Figure 29 shows one such shortened method. Then, too, an extension of this enables the pupils to find the vertices of an inscribed regular dodecagon (Fig. 30).

Octagons, moreover, and squares and sexdecagons appear readily as pupils experiment with circles, halves of circles, halves of halves of circles, halves of halves of halves of circles, etc. (Figs. 31–33). Since the joins of alternate vertices of a sexdecagon form an octagon, and since a square derives in like manner from an octagon, pupils may wonder what happens when they join alternate vertices of

Figure 24

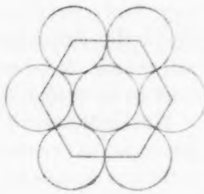


Figure 25

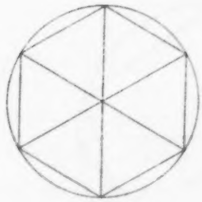


Figure 26

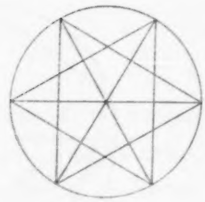


Figure 27

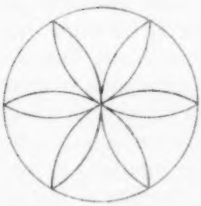


Figure 28

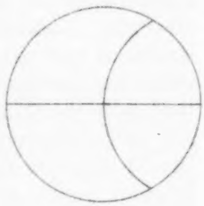


Figure 29

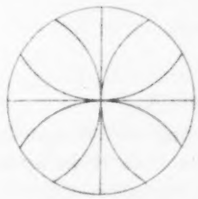


Figure 30

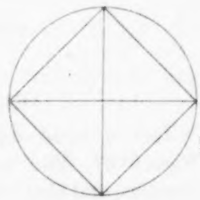


Figure 31

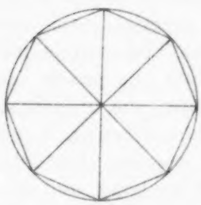


Figure 32

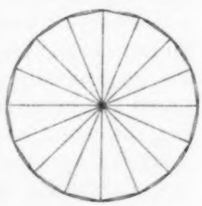


Figure 33

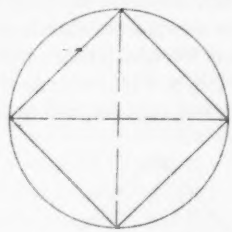


Figure 34

a square. Whether the result is a polygon may provoke discussion (Fig. 34).

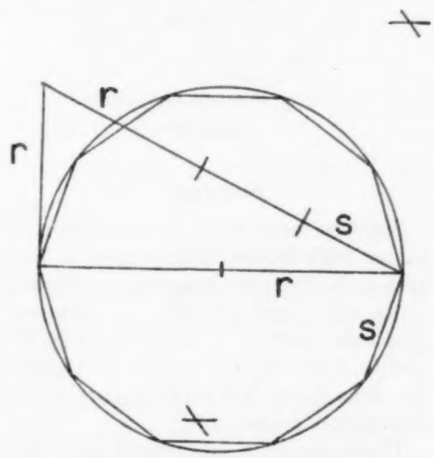
Other regular polygons

How about other regular polygons?

If p is prime, the five smallest values of p for which regular polygons can be constructed with ruler and compasses alone are 3, 5, 17, 257, and 65537. Although this was proved by Gauss at age 17, it hardly fits into geometry in the junior high school. It is not unwelcome information, however, for the teacher whose pupils set out rather blindly to find a way to construct a regular heptagon.

Doubtless the pentagon will also interest some pupils. Figures 35 and 36 represent two ways to construct a regular decagon. The latter figure also contains in it the length of one side of a regular pentagon.

Figure 35



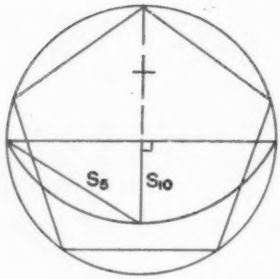


Figure 36

THE DESIGN

Just a word regarding the design in Figure 37. You notice that it involves squares, and that right triangles stand out. Possibilities for discoveries related to the Pythagorean Theorem abound, since each new square stems from the diagonal of its predecessor. Eventually the series 1, $\sqrt{2}$, 2, $2\sqrt{2}$, 4, $4\sqrt{2}$, 8, $8\sqrt{2}$, etc., as the relation among sides of squares, comes to light.

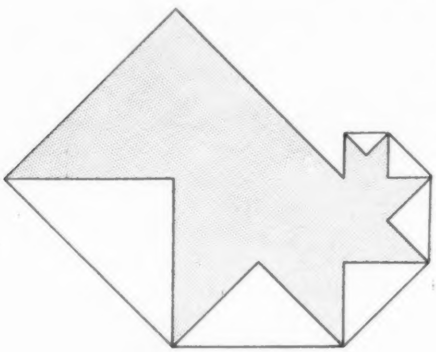


Figure 37

SUMMARY

Suppose, now, that we summarize. In junior high school, pupils need a fresh approach. From kindergarten on, the concepts, rules, and operations of arithmetic dominate children's experiences in mathematics. Few deny this and, if the teaching has been good, still fewer deplore it. But the subject grows stale.

Geometry, then, can provide an untrod way. And even if geometric concepts flourish in the elementary school, new vistas still remain to be seen, and novel domains exist to be explored.

Mathematicians, especially those in the forefront of research, recommend many activities to accelerate children mathematically. In the increasingly widespread zeal to hasten pupils' acquaintance with the mathematics of the past century, an otherwise laudable swing toward abstractions may get out of hand. For, despite their craving for permission to drive the family car, pupils in junior high school are still immature. Concrete geometry should therefore not be omitted; it may heighten pupils' understanding of mathematics and their seeing the need for proof.

The race, including those who advocate abstractions today, learned much of its mathematics through practical needs and by way of real problems. The abstractions, the generalizations, and the proofs followed the experimenting, the investigating, the erring, and the correcting. Surely a small share of such learning while solving, abstracting while applying, and generalizing while testing should not be denied young people today. Nor need this experience deter the potential Norbert Wieners in your classroom.

Some situations wherein pupils can make discoveries for themselves included the making of hollow models of geometric solids, and the drawing, measuring, and altering of plane figures. Of numerous figures that pupils may use to learn geometry, only a few have appeared here. There is almost no limit to the combinations pupils can produce, and the generalizations they can draw from them. Appropriate, too, would be a few projects involving earth measuring. Configurations arising in the study of space travel will also whet pupils' geometric curiosity.

Geometry is the language of form, the science of space. Studying the language of form can become a fascinating quest. Are you ready to face the findings your pupils disclose in this age of space searching?

A test for divisibility

SISTER M. BARBARA STASTNY, O.S.F., *College of St. Joseph on the Rio Grande, Albuquerque, New Mexico.*
Some interesting divisibility properties of numbers.

PRIME NUMBERS have long been an object of considerable interest to mathematicians. Two centuries before Christ, Eratosthenes worked out his sieve for obtaining the prime numbers below a given number. Throughout the centuries following, many interesting studies have been made, and some properties of the primes have been defined.

However, the approach of this paper is almost the reverse of many of these mathematical speculations. In working with applied mathematics, we are often required to determine the prime factors of a composite number. In other cases, we simply wish to know if a composite number is divisible by some certain number, often a prime. The question, then, is how this divisibility can be ascertained without actually carrying out the computation. If the divisor is 2, 3, 4, 5, or 9, the task is easy. All even numbers are divisible by 2; those ending in 0 or 5 are divisible by 5. If the number formed by the last two digits is divisible by 4, the whole number is divisible by 4. After adding the digits, we can test for divisibility by 3 or 9 by determining whether or not the sum is divisible by either number. However, when the divisor is 67 or 43, or even a simpler one such as 7, 11, or 13, we have difficulty. In this paper, I will present one simple and effective way of determining divisibility by these numbers.

Our number system is one that uses a base of ten. Knowledge of this is important in working with numbers. The first example with which we are going to work is based on this fact. We will consider any number as being of the form $(10x+y)$. Thus, 325 is $10(32)+5$, and 273,468 is $10(27346)+8$. Moreover, we know that $(10x+y)$ may be expressed in many ways without changing the value it represents. This is due to the commutative, associative, and distributive laws.

In the following study we are going to "break up" the $10x$ and redistribute the y by adding and subtracting various multiples of it. Thus we can change $(10x+y)$ to $(x+y+9x)$ without changing the value of the integer it represents. If we assume that $(x+y)\equiv 0$, we can determine the modulus by substitution. If $(x+y)\equiv 0 \pmod m$, then $x\equiv -y \pmod m$. By substituting $-y$ for x in the preceding form $(x+y+9x)$, we have $-y+y-9y=-9y$. This is divisible by 9. Hence the modulus in the distribution must be 9. The preceding steps in equation form would be:

$$10x+y=x+y+9x.$$

If $x+y\equiv 0 \pmod m$, then $x\equiv -y \pmod m$, and by substitution

$$10x+y\equiv 0-9y \pmod m.$$

Therefore,

$$10x+y\equiv 0 \pmod 9.$$

What is the numerical application of this? Suppose that we are given a number, any number. We will use 23,418. This number in the form $(10x+y)$ is $10(2341)+8$, where $x=2341$ and $y=8$. By adding x and y, we have 2341 and 8 equal 2349. In exactly the same manner, we can

consider this sum as being of the form $(10x+y)$ and repeat the preceding process. Therefore, $2349=10(234)+9$, where $x_1=234$ and $y_1=9$. Then x_1 plus y_1 is equal to 234 plus 9, or 243. Again, 243 is equal to $10(24)+3$. By letting $x_2=24$ and $y_2=3$, we add x_2 and y_2 to obtain a sum of 27. We could carry this one step further if we wished, but already we can see that 27 is divisible by 9. Therefore, according to this method, the original number should be divisible by 9. Checking this, we find that 23,418 is equal to (2602)(9) and divisibility by 9 is established.

The real value of this system shows itself as we change the arrangement of $(10x+y)$ still more. This time we will consider the form $(x+2y+9x-y)$ and assume that $(x+2y)\equiv 0 \pmod{m'}$; then $x\equiv -2y \pmod{m'}$. Substituting, we have

$$[(x+2y)+9x-y]$$

$$\equiv[(x+2y)-18y-y] \text{ or } [(x+2y)-19y].$$

Since our assumption was that $(x+2y)\equiv 0 \pmod{m'}$, the whole form is congruent to zero, modulus 19, thus ascertaining the value of m'. From this we have now an easy way for determining divisibility of a number by 19, a prime number.

Also, if $(x+2y)\not\equiv 0 \pmod{19}$, then $(10x+y)\not\equiv 0 \pmod{19}$; for if $(10x+y)\equiv 0 \pmod{19}$, then $(20x+2y)\equiv 0$ and $(x+2y)\equiv 0$. But $(x+2y)\not\equiv 0$, so then $(10x+y)\not\equiv 0 \pmod{19}$. That is to say, if $(x+2y)$ is not divisible by 19, then $(10x+y)$ is not divisible by 19.

Two examples will help clarify this. First let us try the number 23,446. In this, $x=2344$ and $y=6$.

$$x+2y=2344+12 \text{ or } 2356.$$

As in the other example, we keep repeating the steps until we arrive at some small number which will tell us if it is divisible by 19. Thus

$$x_1=235 \text{ and } y_1=6$$

$$x_1+2y_1=235+12 \text{ or } 247.$$

Then

$$x_2=24 \text{ and } y_2=7$$

$$x_2+2y_2=24+14 \text{ or } 38.$$

Here we can easily ascertain that 19 divides 38, but we may continue to the end of the sequence, with

$$x_3=3 \text{ and } y_3=8$$

$$x_3+2y_3=3+16 \text{ or } 19.$$

To check this, we divide 23,446 by 19 to obtain the quotient 1234.

Our second numerical example of this type is 923,467. In this example, $x=92,346$ and $y=7$.

$$x+2y=92,346+14 \text{ or } 92,360.$$

Then

$$x_1=9236 \text{ and } y_1=0$$

and

$$x_1+2y_1=9236+0 \text{ or } 9236.$$

From this, we have

$$x_2=923 \text{ and } y_2=6$$

$$x_2+2y_2=923+12 \text{ or } 935.$$

Then

$$x_3=93 \text{ and } y_3=5$$

$$x_3+2y_3=93+10 \text{ or } 103.$$

Once more,

$$x_4=10 \text{ and } y_4=3$$

$$x_4+2y_4=10+6 \text{ or } 16.$$

Now 19 does not divide 16, and therefore it does not divide our original number, 923,467. Checking: $923,467\div 19 \doteq 48,603 + 10$ remainder.

It is significant to note that in the first instance, using $(x+y)$, the modulus was 9, and in the second case, with $(x+2y)$, the modulus was 19. Both end in 9, and the difference is 10. Does this indicate that $(x+3y)$ will yield a modulus of 29? Using the same method of proof as in the earlier cases, we can establish the fact that it does.

As we consider the various moduli from this schema, we find some which are not prime numbers. But any composite is made up of prime numbers. Any number that is divisible by a composite number will also be divisible by the prime factors of the composite number. Thus we extend the list of prime numbers able to be tested by this means. Tabulating the results of these moduli, we have the array shown in Table 1.

TABLE 1

If congruent to zero	*then* $10x+y\equiv 0$ *with modulus*	*and the number is divisible by*
$x+y$	9	1, 3, 9
$x+2y$	19	19
$x+3y$	29	29
$x+4y$	39	3, 13, 39
$x+5y$	49	7, 49
$x+6y$	59	59
$x+7y$	69	23, 69
$x+8y$	79	79
$x+9y$	89	89
$x+10y$	99	3, 9, 11, 99

The next question which may arise is what happens if we use $(x-y)\equiv 0$ instead of $(x+y)\equiv 0$. By exactly the same procedure we have

$$(10x+y)=(x-y)+9x+2y$$

In this case, if $x-y\equiv 0 \pmod{11}$, then $10x+y\equiv 0 \pmod{11}$. The modulus is 11, another prime number. Extending this type of distribution as we did with $(x+y)$, we [illegible]$-2y$). This gives us a modulus of 21. As we continue through the series, we find that $(x-3y)$ yields a modulus of 31; $(x-4y)$, a modulus of 41, and so on through the system. Whenever the form $(10x+y)$ is used to represent the original form of the integer, the moduli increase by ten for each increase in the coefficient of y. Hence, a table of the distribution of $(10x+y)$, using the congruences of an $(x-y)$ arrangement, would show the divisors given in the right-hand column of Table 2.

TABLE 2

If congruent to zero	*then* $10x+y\equiv 0$ *with modulus*	*and the number is divisible by*
$x-y$	11	11
$x-2y$	21	3, 7, 21
$x-3y$	31	31
$x-4y$	41	41
$x-5y$	51	3, 17, 51
$x-6y$	61	61
$x-7y$	71	71
$x-8y$	81	9, 27, 81
$x-9y$	91	3, 13, 91
$x-10y$	101	101

A numerical example of divisibility using $(x-10y)$ is now given. Using the modulus given in the table, we will determine whether 9,843,216 is divisible by 101. I will arrange the work vertically this time but will follow the same procedure used in the other examples. Actually I prefer the vertical arrangment, as it appears more concise and involves less frequent repetition of numbers.

$$10x+y=10(984321)+6$$

$$\begin{array}{lr}
x-10y & 9843216 \\
 & -60 \\ \hline
x_1-10y_1 & 984261 \\
 & -10 \\ \hline
x_2-10y_2 & 98416 \\
 & -60 \\ \hline
x_3-10y_3 & 9781 \\
 & -10 \\ \hline
x_4-10y_4 & 968 \\
 & -80 \\ \hline
 & 16
\end{array}$$

Therefore, we see that the number is not divisible by 101, since 16 is not divisible by it.

A further extension of this basic idea is to consider the integer as being of the form $(100x+y)$ instead of $(10x+y)$. Numerically then, $734{,}862=100(7348)+62$, where $x=7348$ and $y=62$.

By redistributing this we may have $(x+y+99x)$. Then, by letting $(x+y)\equiv 0$ (mod m), $x\equiv -y$ (mod m). After substitution, $(100x+y)\equiv(0+99y)$ mod m. Using a modulus of 99, the whole integer is congruent to zero.

When $(100x+y)=(x+2y+99x-y)$, we determine that the modulus is 199. This follows the same pattern as the preceding arrangements of the moduli in the $(10x+y)$ form, with the modulus now increasing 100 each time that the coefficient of y increases in the congruence. Consequently, we can set up Table 3.

TABLE 3

If congruent to zero	*then* $100x+y\equiv 0$ *with modulus*	*and the number is divisible by*
$x+y$	99	3, 9, 11
$x+2y$	199	199
$x+3y$	299	13, 23, 299
$x+4y$	399	2, 7, 19
$x+5y$	499	499
$x+6y$	599	599
$x+7y$	699	3, 233
$x+8y$	799	17, 47
$x+9y$	899	29, 31
$x+10y$	999	3, 9, 37

Another possibility is to convert $(100x+y)$ to a form $(x-y)+99y+2y$. Here the necessary modulus is 101, if the whole integer is to be congruent to zero. As the $(x-y)$ is changed to $(x-2y)$, the modulus increases to 201. As we see, this will again follow the previous array of increasing moduli (Table 4).

TABLE 4

If congruent to zero	*then* $100x+y\equiv 0$ *with modulus*	*and the number is divisible by*
$x-y$	101	101
$x-2y$	201	3, 67
$x-3y$	301	7, 43
$x-4y$	401	401
$x-5y$	501	3, 167
$x-6y$	601	601
$x-7y$	701	701
$x-8y$	801	3, 9, 89
$x-9y$	901	17, 53
$x-10y$	1001	7, 11, 13

By using these last two tables, we have a convenient method of testing the divisibility of many numbers. Moreover, when we use $(100x+y)$ the number of steps required to "reduce" a number is decreased.

Many of the numbers that were found on the first two tables are repeated in these last two tables. This is of great advantage, for we can test divisibility of several numbers simultaneously. For example, by using $(x-10y)$, we are able to tell, with only one series of operations, whether a number is divisible by 7, 11, and 13. The work for this calculation is given.

Given: the integer 13,653,926.

$$100x+y=100(136539)+26$$

$$\begin{array}{lr}
(x-10y) & 13653926 \\
 & -260 \\
\hline
 & 136279 \\
 & -790 \\
\hline
 & 572 \\
 & -720 \\
\hline
 & 715 \\
 & -150 \\
\hline
 & 143
\end{array}$$

Since 143 is divisible by 11 and 13, but not by 7, the integer 13,653,926 is divisible by 11 and 13 but is not divisible by 7. Actual division confirms this. The reader may wish to prove the theorems needed to draw these last conclusions.

This example presents two other points which should be noted: (1) that signs are not an influencing element in the calculations, and (2) that the remainder, when the "reduced number" is divided, is not the numerical value of the true remainder of the given integer. In the above problem, we had the number 572, where $x=5$ and $10y=720$. By subtracting $10y$ from x, we had a negative number, -715, remaining. When proceeding to the next step, we used $x=7$ and $10y=150$ without reference to the negative sign. The number may be

"reduced" as far as we desire to obtain a convenient test, without bothering about signs but using only the absolute values.

For very large numbers, the conversion to the form $(1000x+y)$ has some definite advantages. The schema of the moduli would follow the same form as those of the preceding examples, with the modulus for $(x+y)$ being 999; for $(x+2y)$, 1999; for $(x+3y)$, 2999, and so on. When we use $(x-y)$, the modulus is 1001. For $(x-2y)$, it is 2001, and it continues as the other patterns have, the only exception being that each increases 1000 this time.

When we use $(x-10y)$ in the integral form $(1000x+y)$, the modulus is 10,001. This gives a means of determining divisibility by 73, a prime number not included in the other tables. If a check is made of the tabulated results, it may be seen that all the prime numbers below 100 are accounted for, except 73, 83, and 97. By using the $(1000x+y)$, we have a test for 73. The numbers 83 and 97 do not appear in any of the moduli below 10,001.

The work begun in this paper opens many avenues for investigation. One would be to use various systems of numbers having a base other than 10. Possibly in this way we could find a means of testing for 83 and 97. A more involved search may show some relations among the remainders of the given integer and the "reduced number." All these show promise of further properties existing among the numbers of our decimal system.

What's new?

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Recommendations of the Mathematical Association of America for the training of teachers of mathematics

THE COMMITTEE on the Undergraduate Program in Mathematics (CUPM)[1] is a committee of the Mathematical Association of America and is supported in part by the National Science Foundation. The general purpose of this committee is to develop a broad program of improvement in the undergraduate mathematics curriculum of the nation's colleges and universities.

As part of its mandate, CUPM established a Panel on Teacher Training.[2] This panel was instructed to prepare for CUPM a set of recommendations of minimum standards for the training of teachers on all levels. The following report is the result of the work of the Panel on Teacher Training and has received the endorsement of the Committee on the Undergraduate Program in Mathematics and of the Board of Governors of the Mathematical Association of America.

The Panel on Teacher Training has been further charged with the implementation of these recommendations and hopes to issue supplementary reports, as well as to hold various regional conferences, to make these minimum standards a reality.[3]

The report consists of the following: (1) general recommendations, (2) the five levels, (3) recommendations for the five levels, (4) summary of recommendations, (5) curriculum-study courses, (6) training of supervisors, and (7) sample course descriptions.

GENERAL RECOMMENDATIONS

The purpose of this report is to present a preliminary outline of the panel's recommendations for the minimal college-training program for teachers of mathematics. We have found it a most useful device to arrive at a classification of mathematics teachers which does not, as far as we know, depend on any present scheme of training teachers for their various tasks. The existing classifications seem to have arisen from a series of historical accidents and from fundamental psychological considerations. We hope this report reflects our feeling that we have made a serious attempt to classify teachers according to their position in an over-all sequential schedule of presenting the main ideas of mathematics.

For each classification presented, we give a recommendation as to the type and minimum amount of mathematics which should be taken by the student preparing for a career in teaching. Further, we spell

[1] Members: R. C. Buck, *Chairman*, R. J. Wisner, *Executive Director*, E. G. Begle, L. W. Cohen, W. T. Guy, Jr., R. D. James, J. L. Kelley, J. G. Kemeny, E. E. Moise, J. C. Moore, Frederick Mosteller, H. O. Pollak, G. B. Price, Patrick Suppes, Henry Van Engen, R. J. Walker, and A. D. Wallace.

[2] Members of the Panel on Teacher Training: John G. Kemeny, Dartmouth College, *Chairman*; E. G. Begle, Yale University; W. T. Guy, Jr., University of Texas; P. S. Jones, University of Michigan; J. L. Kelley, University of California, Berkeley; Bruce Meserve, Montclair State College; E. E. Moise, Harvard University; Rothwell Stephens, Knox College; Henry Van Engen, University of Wisconsin. Ex officio members of the panel: R. Creighton Buck, University of Wisconsin, and Robert J. Wisner, Michigan State University Oakland.

[3] Reprints and further information may be obtained by writing the Executive Director, Professor Robert J. Wisner, Michigan State University Oakland, Rochester, Michigan.

out in some detail the *types* of courses—included as an Appendix—which we recommend to implement the goals described. The courses we describe which are specifically designed for prospective teachers should be taught by persons who are masters of their subject matter and who have, in addition, a knowledge of the problems which teachers face.

These sample courses are given solely for illustrative purposes to explain the type of courses and the levels of advancement desirable for prospective teachers. It should be clearly understood that different institutions will wish to exercise considerable freedom in implementing these recommendations, both as to the way topics are combined into courses and as to the exact choice of topics for individual courses.

There are several very sincere words of warning to be put forth in regard to the reading and interpretation of this report.

First, the classifications are to be taken in the rather loose fashion in which they are described. Their exact delineations will of course depend upon local conditions of school and curricular organization. It should be noted that the various classifications overlap: this is done deliberately in an attempt to meet just such local conditions.

Second, the recommendations are not motivated by a desire to meet the demands of any special program of mathematics education; nor do the descriptions or outlines of courses to be taken by prospective teachers represent an attempt on the part of this committee to further the goals of any particular school curriculum-planning organization. The recommendations are meant to be the minimum which should be required of teachers in any reasonable educational program, and the course descriptions are presented only to illustrate what is meant by the course titles.

Third, it is to be hoped that everyone recognizes good mathematics education to be a sequential experience. Thus, the teacher at any particular level should have an understanding of the mathematics which will confront the student in subsequent courses; and as a consequence, it is desirable that a teacher at a given level be prepared to teach at least some succeeding courses. Ideally, a person preparing for teaching should meet, in addition to the minimal requirement set forth here, as many of the requirements for the next level as his or her college program permits.

Fourth, this report is meant as a guide to the preparation of people who will be teaching any mathematics whatsoever. The suggestions apply, within each level, to all people who teach any mathematics. The recommendations do not in any sense exclude the teacher who is assigned classes scheduled primarily for students of low aptitude.

Fifth, every good teacher knows that mathematics must begin at the concrete level before it can proceed to the more technical or abstract formulation. Motivation for new concepts must be derived, and later application of the theory to nature must be included. In each of the outlines to follow, it is assumed that topics will contain a judicious mixture of motivation, theory, and application. A purely abstract course for teachers would be madness, and a course in calculation with no theory would not be mathematics.

Sixth, the phrase "a course" occurs in several places in this report. For purposes of fixing ideas, this phrase is employed in the sense of a three-semester-hour presentation of the subject matter described, and it is not meant to exclude integrated programs or other curricular arrangements.

Finally, the reader should note that the training for Level I teaching is a separate program, while the curricula for the further levels form a cumulative sequence, in which each program is a continuation of the preceding one.

The committee benefited greatly from previous studies on teacher preparation,

such as that of the Cooperative Committee on the Teaching of Science and Mathematics, a committee of the American Association for the Advancement of Science. The committee was also guided by discussions with a variety of professional organizations. It is pleased to note the considerable degree of agreement common to all proposals.

It should be emphasized that these recommendations are minimal in nature and that some institutions have already met and exceeded these recommendations. It is expected that as high school curricula are strengthened, these minimum recommendations will be revised.

The five levels

I. *Teachers of elementary school mathematics.* This level consists of teachers confronted with the problem of presenting the elements of arithmetic and the associated material now commonly taught in grades K through 6. The committee recognizes that special pedagogical problems may be connected with grades K through 2, and so a special program may be appropriate for teachers of such grades.

II. *Teachers of the elements of algebra and geometry.* Included here are teachers who are assigned the task of giving introductory-year courses in either algebra or geometry, or the less formal preliminary material in these fields. These introductory courses are now commonly taught in grades 7 through 10.

III. *Teachers of high school mathematics.* These teachers are qualified to teach a modern high school mathematics sequence[4] in grades 9 through 12.

IV. *Teachers of the elements of calculus, linear algebra, probability, etc.* This is a mixed level, consisting of teachers of advanced programs in high school, junior college teachers, and staff members employed by universities to teach in the first two years. These teachers should be qualified to present a modern two-year college mathematics program.

V. *Teachers of college mathematics.* These teachers should be qualified to teach all basic courses offered in a strong undergraduate college curriculum.

The levels having been presented, we are now ready to proceed to a description of our recommendations of the minimal college training requirements for entry into the teaching profession at each level.

Recommendations for level I

As a prerequisite for the college training of elementary school teachers, we recommend at least two years of college preparatory mathematics, consisting of a year of algebra and a year of geometry, or the same material in integrated courses. It must also be assured that these teachers are competent in the basic techniques of arithmetic. The exact length of the training program will depend on the strength of their preparation. For their college training, we recommend the equivalent of:

A. A two-course sequence devoted to the structure of the real number system and its subsystems (See Course-sequence 1).[5]

B. A course devoted to the basic concepts of algebra (See Course 2).

C. A course in informal geometry (See Course 3).

The material in these courses might, in a sense, duplicate material studied in high school by the prospective teacher, but we urge that this material be covered again, this time from a more sophisticated, college-level point of view. Whether the material suggested in A above can be covered in one or two courses will clearly depend upon the previous preparation of the student.

We strongly recommend that at least 20 per cent of the Level I teachers in each

[4] Such sequences have been recommended by the Commission on Mathematics, the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, and others.

[5] Sample courses, by numbers, are to be found in the Appendix.

school have stronger preparation in mathematics, comparable to Level II preparation but not necessarily including calculus. Such teachers would clearly strengthen the elementary program by their very presence within the school faculty. This additional preparation is certainly required for elementary teachers who are called upon to teach an introduction to algebra or geometry.

Recommendations for level II

Prospective teachers of the elements of algebra and geometry should enter this program ready for a mathematics course at the level of a beginning course in analytic geometry and calculus (requiring a minimum of three years in college preparatory mathematics). It is recognized that many students will have to correct high school deficiencies in college. (However, such courses as trigonometry and college algebra should not count toward the fulfillment of minimum requirements at the college level.) Their college mathematics training should then include:

A. Three courses in elementary analysis (including or presupposing the fundamentals of analytic geometry). (See Course-sequence 4.)

This introduction to analysis should stress basic concepts. However, prospective teachers should be qualified to take more advanced mathematics courses requiring a year of the calculus, and hence calculus courses especially designed for teachers are normally not desirable.

B. Four other courses: a course in abstract algebra, a course in geometry, a course in probability from a set-theoretic point of view, and one elective. One of these courses should contain an introduction to the language of logic and sets. (See Courses 5–7.)

Recommendations for level III

Prospective teachers of high school mathematics beyond the elements of algebra and geometry should complete a major in mathematics and a minor in some field in which a substantial amount of mathematics is used. This latter should be selected from the areas in the physical sciences, biological sciences, and the social studies, but the minor should in each case be pursued to the extent that the student will have encountered substantial applications of mathematics.

The major in mathematics should include, in addition to the work listed under Level II, at least an additional course in each of algebra, geometry, and probability-statistics, together with two electives.

Thus, the minimum requirements for high school mathematics teachers should consist of the following:[6]

A. Three courses in analysis (See Course-sequence 4).

B. Two courses in abstract algebra (See Course-sequence 5).

C. Two courses in geometry beyond analytic geometry (See Course-sequence 6).

D. Two courses in probability and statistics (See Course-sequence 7).

E. Two upper-class elective courses, e.g., introduction to real variables, number theory, topology, history of mathematics, or numerical analysis (including use of high-speed computing machines).

One of these courses should contain an introduction to the language of logic and sets, which can be used in a variety of courses.

Recommendations for level IV

For teachers of the elements of calculus, linear algebra, probability, etc., we recommend a Master's degree, with at least two thirds of the courses being in mathematics, and for which an undergraduate program at least as strong as Level III training is a prerequisite. A teacher who has completed the recommendations for Level III should use the additional mathe-

[6] The requirements for Level II preparation have been included in this list.

matics courses to acquire greater mathematical breadth.

Since these teachers will be called upon to teach calculus, we recommend that the program include the equivalent of at least two courses of theoretical analysis in the spirit of the theory of functions of real and complex variables.

It is important that universities have graduate programs available which can be entered with Level III preparation, recognizing that these students substitute greater breadth for lack of depth in analysis as compared with an ordinary B.A. with a major in mathematics. In other respects, graduate schools should have great freedom in designing the M.A. program for teachers.

Recommendations for level V

We recognize the tremendous problems created by the shortage of qualified college mathematics teachers. A recommendation for the alleviation of this problem is now receiving serious attention.

Summary of recommendations

The recommendations as to the minimum amount and type of mathematics which should be taken by the prospective teacher are summarized in Tables 1 and 2.

Curriculum-study courses

The foregoing recommendations have dealt in detail with the subject-matter training of mathematics teachers. There are many other facets to the education of the scholarly, vigorous, and enthusiastic persons to whom we wish to entrust the education of our youth. One of these merits special mention by us. Effective mathematics teachers must be familiar with such items as:

A. The objectives and content of the many proposals for change in our curriculum and texts.

TABLE 1

Minimum standards for training of mathematics teachers

Level	*Description*	*Degree*	*High school prerequisites*	*Minimum number of college courses*
I	Elementary school	B.A.	Two years of college preparatory mathematics	4
II	Elements of algebra and geometry	B.A., math. minor	Preparation for analytic geometry and calculus	7
III	High school	B.A., math. major	Preparation for analytic geometry and calculus	11
IV	Elements of calculus, linear algebra, probability, etc.	M.A. in math.	Preparation for analytic geometry and calculus	Approx. 18

TABLE 2

Breakdown by subjects

Level	*Numbers*	*Analysis*	*Algebra*	*Geometry**	*Probability-statistics*	*Elective*
I	2		1	1		
II		2	1	2	1†	1‡
III		2	2	3	2†	2‡
IV§		4	2	3	2	7

* Including analytic geometry.
† An introduction to the language of logic and sets should appear in some one course.
‡ Preferably from the areas specified.
§ The numbers in this row indicate the approximate number of courses.

B. The techniques, relative merits, and roles of such teaching procedures as the inductive and deductive approaches to new ideas.
C. The literature of mathematics and its teaching.
D. The underlying ideas of elementary mathematics and the manner in which they may provide a rational basis for teaching, unless taken care of by mathematics courses especially designed for teachers.
E. The chief applications which have given rise to various mathemetical subjects. These applications will depend upon the level of mathematics to be taught and are an essential part of the equipment of all mathematics teachers.

Such topics are properly taught in so-called "methods" courses. We would like to stress that adequate teaching of these can be done only by persons who are well informed *both* as to the basic mathematical concepts and as to the nature of American public schools, and as to the concepts, problems, and literature of mathematics education. In particular, we do not feel that this can be done effectively at either the elementary or secondary level in the context of "general" methods courses, or by persons who have not had at least the training of Level IV.

Training of supervisors

There is a great need for providing adequately trained supervisors of mathematics, grades K–12, for our public schools. At present, administrators find no ready supply of such individuals and, hence, are through necessity making appointments which are highly questionable, if not indefensible. For this reason, it is urgent to develop a program for supervisors and to seek adequate support for those individuals who have the desired qualities for supervision and the ability to benefit from advanced training. Such training would prepare the "leaders of teachers" in the local system (1) to make sound judgments concerning mathematics programs for the schools, (2) to understand thoroughly the recommendations made by national committees, and (3) to enable schools better to articulate school mathematics with college mathematics.

Prerequisite to this program should be a regular Master's degree in mathematics or a Master's degree given as a result of participation in an Academic Year Institute. The program should consist of additional graduate courses in abstract algebra, analysis, and geometry, with courses selected from logic, statistics, theory of numbers, philosophy of education, history of education, history of mathematics, seminar courses on the program of the elementary school and secondary school mathematics, and additional electives in algebra, analysis, or geometry to provide some degree of concentration.

The committee feels that action must be taken to fill the need for supervisory personnel, and we recommend such action to the appropriate authorities.

APPENDIX: Course descriptions

We list below sample courses that might be used to fulfill the minimum requirements for Levels I through III, and the undergraduate requirements of Levels IV and V. These brief descriptions are included to clarify the meaning of course titles but are not intended as syllabi for actual courses. It must be recognized that there are other equally good ways of combining various recommended topics, and colleges should be encouraged to work out detailed curricula to suit their own tastes and local conditions. However, the committee hopes that these very brief descriptions will help in indicating the types of courses desirable and the level of advancement.

For level I

1. *Algebraic structure of the number system* (2-course sequence).

This is a study of the numbers used in elementary school: whole numbers, common fractions, decimal fractions, irrational numbers. Emphasis should be on the basic concepts and techniques: properties of addition, multiplication, inverses, systems of numeration, and the number line. The techniques for computation with numbers should be derived from the properties and structure of the number system, and some attention should be paid to approximation. Some elementary number theory, including prime numbers,

properties of even and odd numbers, and some arithmetic with congruences should be included.

2. *Algebra.*

   Basic ideas and structure of algebra, including equations, inequalities, positive and negative numbers, absolute value, graphing of truth sets of equations and inequalities, examples of other algebraic systems—definitely including finite ones—to emphasize the structure of algebra as well as simple concepts and language of sets.

3. *Intuitive foundations of geometry.*

   A study of space, plane, and line as sets of points, considering separation properties and simple closed curves; the triangle, rectangle, circle, sphere, and other figures in the plane and space considered as sets of points with their properties developed intuitively; the concept of deduction and the beginning of deductive theory based on the properties that have been identified in the intuitive development; concepts of measurement in the plane and space, angle measurement, measurement of the circle, volumes of familiar solids; treatment of co-ordinate geometry through graphs of simple equations.

FOR LEVELS II–V

4. *Analytic geometry and calculus* (3-course sequence).

   Approximately one-third of the sequence should be devoted to analytic geometry, taught either in co-ordination with calculus or after the calculus sequence. This should include the co-ordinate plane, functions, polar co-ordinates, the algebraic description of subsets of the plane—related to solutions of equations—and parametrically as the range of a function, change of co-ordinates, and brief treatment of conic sections.

   The sequence should also give a thorough treatment of the calculus for functions of one variable, with stress on the basic ideas, but with adequate attention to manipulative skills. The course should introduce differentiation, integration, the rational, trigonometric, and exponential functions, as well as a brief treatment of series and some very elementary differential equations.

5. *Abstract algebra* (2-course sequence).

   One course in this sequence constitutes an introduction to algebraic structures, such as groups, rings, fields, etc. The basic approach is to proceed from the concrete to the abstract. Use should be made of algebraic systems familiar to the student in order to motivate the abstract axioms. On the one hand, stress should be placed on rigorous algebraic proofs to convince the student that geometry is not the only area for axiomatic treatment. On the other hand, to keep the abstract procedure tied to the student's experience, various "concrete" applications should be given for theorems. Examples should be drawn from number systems, geometry, and other areas.

   The other course should be devoted to linear algebra, restricted to real, finite-dimensional cases. This can be introduced by concrete manipulation of vectors and matrices, after which the student should be motivated to free himself from the accident of the choice of a basis. The student should be taught the handling of vector equations and inequalities along with an intuitive introduction to linear programming and games. A good treatment of linear functions and transformations is needed, including a thorough understanding of the solution of m equations in n unknowns.

6. *Geometry* (2-course sequence).

   These recommendations have in general been based on the idea that advanced courses for teachers should be designed in such a way as to deepen understanding of the material which they will be teaching. In geometry, such a program involves special problems, because here some of the appropriate background material is not ordinarily thought of as being geometry at all, and much of it is not ordinarily taught on the undergraduate level.

   The foundations of geometry, in the sense of Hilbert, is only one among many topics. Some further examples are as follows:

   a. Generalization of the idea of congruence to include rigid motions, that is, one-to-one correspondences preserving distances.
   b. A corresponding generalization of the idea of similarity.
   c. Enough measure theory to turn the familiar area and volume formulas into theorems, and to justify Cavalieri's Principle.
   d. "Pure analytic geometry," in which points, lines, and so on are defined and treated in terms of a co-ordinate system, without the use of any synthetic postulates at all. This is quite different from conventional analytic geometry in which the synthetic postulates are used in the very construction of co-ordinate systems. The "purely analytic" treatment can be used to give a consistency proof for the synthetic postulates.

   These topics are given merely as illustrations of the sort of material that is needed. The choice of topics and the order of priority may require considerable study. The course might well take the form of a series of fairly long digressions from an outline of a high school course, with each advanced topic taken up at the point where it seems most relevant.

7. *Probability and statistics* (2-course sequence).

   The purpose of this sequence is to introduce the student to probability theory from a set-theoretic point of view, and to apply basic

(Continued on p. 643)

Divisibility by two

KENNETH B. PARSONS *and* STANLEY P. FRANKLIN,
University of California, Los Angeles, California.
An interesting theorem relating properties of odd and even integers to number notation in bases other than ten.

NORMALLY questions of divisibility must await the development of a theory of arithmetic congruences for their solution. However, the special case of divisibility by two for numbers written in symbols using arbitrary number bases b (b odd or even), can be decided by very elementary means which are well within the range of many second year high school algebra students. The demonstration follows from easily verified properties of odd and even integers and could provide a valuable example of a mathematical proof. Also, the ideas involved could form the basis of several exercises in the use of number bases other than ten. Students could show, for example, that 212_{three} having an odd number (1) of odd digits, is odd, while 596367_{eleven}, having an even number (4) of odd digits, is even.

For later use we list five properties of odd and even integers which follow readily from any definition of odd and even agreeing with our intuitive notions.

1. If a is even and b any integer, $a \cdot b$ is even.
2. If a and b are odd, $a \cdot b$ is odd.
3. The sum of any finite number of even integers is even.
4. The sum of n odd integers is even or odd according as n is even or odd.
5. If a is odd and b is even, then $a+b$ is odd.

THEOREM I: Given a number N written in base b, where b is odd, N is odd or even as the number, n, of odd digits in its expression is odd or even.

Proof: Let

$$N = \sum_i a_i b^i$$

where the a_i are non-negative integers. Let $J = \{a_j\}$ be the set of all even digits a_j and $K = \{a_k\}$ be the set of all odd digits a_k. Then by (1) and (3)

$$\sum_J a_j b^j$$

is even, and by (2) and (4)

$$\sum_K a_k b^k$$

is odd or even as n is odd or even. Hence by (3) and (5) the theorem follows.

For completeness, we include the following somewhat trivial theorem.

THEOREM II: Given a number N written in base b, where b is even, N is odd or even as its units digit is odd or even.

Proof: Let

$$N = \sum_{i=0}^{n} a_i b^i$$

where the a_i values are non-negative integers. By (1) and (3)

$$\sum_{i=1}^{n} a_i b^i$$

is even. Hence by (3) and (5) N is odd or even as a_0 is odd or even.

Edited by Eugene D. Nichols, Florida State University, Tallahassee, Florida

Seventh graders volunteer for after-school classes in algebra

by Henry Standish, West Genesee Junior High School, Camillus, New York

You may believe that a teacher would volunteer to stay after school to teach algebra, but do you think that he could get students to volunteer to come and learn after the other children have gone home? I have been doing just this since last September and find great pleasure in it. Of course, many people do not understand why a teacher should take on an extra burden without pay. The father of one of my students asked me, at a P.T.A. meeting, whether I got paid for this extra work and could hardly believe that I received no financial reward. Although I am not opposed to monetary reward, I feel highly rewarded by the children themselves. Their enthusiasm and eagerness is a joy to me. Yet I do not think that I could keep the children's interest if it were not for the Madison Project.

The Madison Project teaches algebra, a ninth grade subject, to seventh or lower grade students. I believe that children enjoy doing work beyond their grade because it gives them a feeling of importance; they are proud to do advanced work.

The first I learned of the Madison Project was at the mathematics teachers convention in May 1959 at Syracuse. Professor Robert B. Davis here presented an entirely new approach to teaching algebra. I was fascinated by this method, and when a course was offered at Syracuse University last fall, I registered for it.

Usually the Madison Project is taught twice a week during regular classroom periods. However, the seventh graders at our school are on half sessions until the new Junior High School is ready in September 1960. Half the students attend classes from 8:10 to 12:10 and the other half from 12:22 to 4:25. There are only four periods of instruction. Under these circumstances I did not feel justified to use the regular period for the Madison Project. Therefore, with the principal's permission, I asked my students if they would be interested in joining a Math Club once a week after school. Anyone who had a passing grade could join, and any student who did not like it could drop out at any time. There would be no credit and no examinations. The only requirement was regular attendance; homework was strictly on a voluntary basis.

We are a centralized school and almost all children come by bus. Since the last buses leave at 4:30 P.M., it would be necessary for the parents to pick up their children at 5:15 P.M. This, of course, created a difficult situation and I did not expect many volunteers. Yet I had a group of about thirty at the first meeting.

Now the time had come to teach the

Madison Project. At our first meeting, as I stood in front of the class, I wondered: Will the pupils react as I was told they would, i.e., will they find the answers to problems without being told? I wrote on the blackboard:

$$3+\square=7.$$

When I turned around several hands were raised, and the correct answer was given. After a few minutes of practicing this kind of equation, everyone could supply correct answers. Then I wrote

$$3+(2\times\square)=7.$$

To my knowledge, none of the children had ever used parentheses, and yet the correct answer was forthcoming at once. True, not all the children could find the truth set for this equation, but after a few more of the same type, all seemed to know the correct use of parentheses. Strangely enough, no one asked why the parentheses were used. The children just knew how to use them.

I found that the children did exactly what I was told they would do. They found their own way to a solution. This is part of the Madison Project method, and the teacher is not to tell the students how to solve a problem. Here is what Professor Davis, the author of the Madison Project, says: "We almost never tell students what to do, nor how to do it. Instead, we ask questions."

We then played the matrix game, which involves adding positive and negative numbers. The students form two groups, A and B. (The rivalry can be greatly increased if the boys play against the girls.) A matrix is written on the blackboard:

$$\begin{array}{ll|c}
 & & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \quad \text{Group A's choices} \\
\hline
\text{Group} & 1 & \\
\text{B's} & 2 & \begin{pmatrix} +1 & -1 & 0 \\ +4 & -3 & -1 \\ -5 & +4 & +1 \end{pmatrix} \\
\text{choices} & 3 &
\end{array}$$

Group A secretly selects a letter and writes it on a slip of paper; Group B selects a number. The papers are handed to a student at the board and he finds the number corresponding to their choices. The first choice was "B 3" and +4 was found to be the corresponding number. Group A had won four points. The next choice was "A 3" which meant −5. Group A had lost 5 points. But what was the score now? How could you subtract 5 from 4? These children had never dealt with negative numbers. Would they be able to add +4 and −5? Most of them were puzzled. They all tried to find the score, and then one girl in Group A blurted out: "Now we are one in the hole." This was her way of saying "the score is −1." After this we could play the game quickly and more and more students were able to give the correct score. Before long, most of the children could add positive and negative numbers.

Then we learned about equations in two variables, like the following:

$$\triangle=(4\times\square)+1$$

and made a truth set for it, like this:

$\square$	$\triangle$
1	5
2	9
3	13
4	17
5	21

I was amazed how fast the children would call the next number for the truth set. As I was writing the numbers on the board, the answers were called from all parts of the classroom. We then proceeded to make graphs for the equations. I had prepared a 4′×4′ board with 1½″ squares. On this we marked the axes, as in Figure 1, and started plotting the points. As we were graphing several equations, someone remarked that all points for every equation were on a straight line. The next step was the discovery of a pattern, or different slope. After we had plotted two points of an equation, a boy called: "The next

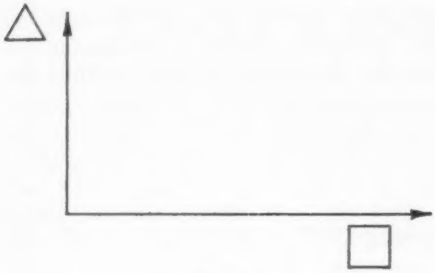


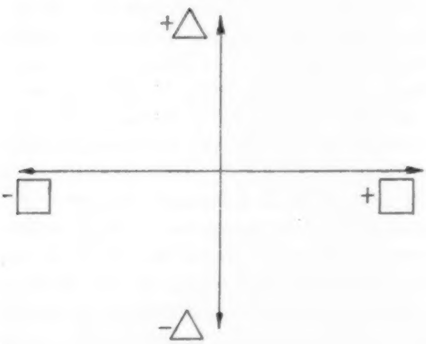
Figure 1

point is one over and four up," and so it was all the way. One student would make the discovery and then, after some practice, almost all students would understand the problem. In the beginning I was skeptical about the children making the discoveries and finding the rules. However, they do find a way to solve the problems and are eager to use short cuts.

Some time later, in order to practice plotting points and to combine it with the use of positive and negative numbers, we extended the axes to the left and down (Fig. 2). To make it interesting, we played battleship. Before the class, one girl and one boy had marked on graph paper where their ships were located.

Four points were used for a battleship, three points for a cruiser, and two for a destroyer (Fig. 3). The boys played against the girls. On the graph board I had marked two areas, one for the boys

Figure 2



and one for the girls, corresponding to the plans previously made but without the plotted points. The groups took alternate turns in plotting points. Two students were at the board and marked each point as it was called, so that the children would not call the same point again. Each time the leader with the plan had to say whether or not it was a hit. When a hit was announced, the excitement was great and the children searched near the hit for the other points of the ship. It was remarkable how well the children could name the location of points just by using

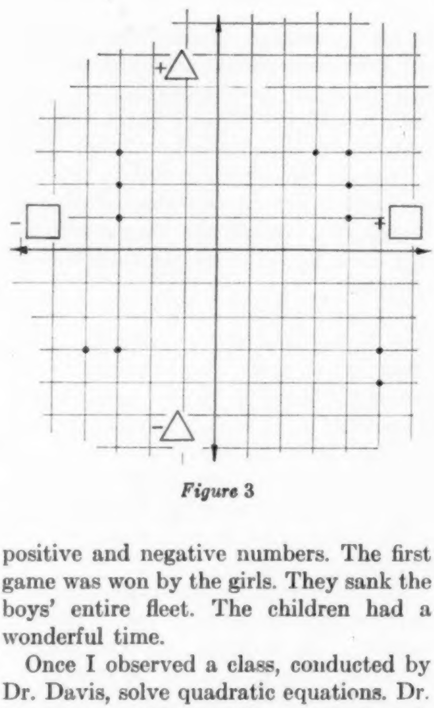


Figure 3

positive and negative numbers. The first game was won by the girls. They sank the boys' entire fleet. The children had a wonderful time.

Once I observed a class, conducted by Dr. Davis, solve quadratic equations. Dr. Davis wrote this equation on the board:

$$(\square\times\square)-(13\times\square)+22=0$$

and indicated that the truth set contained two numbers. The first few equations were solved by trial and error. But soon two boys must have noticed that $2+11=13$ and $2\times11=22$, and that this was true for the two numbers in every equa-

tion, because suddenly they would give answers almost immediately after the equation had been written on the board. They had found the "secret." It was fascinating to observe how more and more children found the secret and after about 25 minutes almost everyone had made the discovery. At no time did Dr. Davis say what the secret was. When I took up quadratic equations in my class, the children were so eager and excited, that they asked for more and more equations without noticing that we went beyond the time limit of our class, and this is really rare with seventh graders.

The Madison Project uses the discovery method. The teacher does not tell the students what to do, but rather asks questions. The children find the answers on their own. They have to think. There is no rote learning or memorizing, and the students understand what they are doing. But most amazing is their enthusiasm.

After a few sessions, more children asked to join and our group grew to about 40. Children from other classes, whom I did not even know, heard about the club and came to ask permission to join. Several children from the morning group came back to school at 4:30 p.m. to participate. The attendance fluctuates but we always have 30 to 35 students.

The Madison Project created an interest and enthusiasm which I have never experienced before. I am sure that these youngsters will continue to study mathematics beyond the minimum requirement. Although children usually do not like to spend an extra hour at school, these youngsters stay of their own free will and enjoy it. The Madison Project proves that children can be made to like mathematics.

(*Continued from p.* 638)

Recommendations for the training of mathematics teachers

probability theory to problems of statistical inference. The first course should be an introduction to random variables on a finite space. It must include motivation, axiomatic treatment of a measure on a finite space, and the proof of a few key theorems. There should be numerous applications from elementary statistics, stochastic processes, and everyday life.

In the second course more stress should be placed on stochastic processes, and probabilities on a continuous sample space should be treated. A substantial amount of time could be devoted to the development of principles of statistical inference.

Note: One of the course sequences, 4 through 7, should include an introduction to the language of logic and sets, so that these concepts may be used wherever appropriate. This introduction could be restricted to a brief treatment of the propositional calculus and of Boolean algebra, stressing the isomorphism between the two structures.

ELECTIVES (FOR LEVELS II–V)

8. *Introduction to real variables.*

9. *Number theory.*

10. *Elementary topology.*

11. *History of mathematics.*

12. *Numerical analysis, with the use of machines*

Edited by Howard Eves, University of Maine, Orono, Maine

The evolution of extended decimal approximations to π

by J. W. Wrench, Jr., Applied Mathematics Laboratory, David Taylor Model Basin, Washington, D.C.

In his historical survey of the classic problem of "squaring the circle," Professor E. W. Hobson [1]* distinguished three distinct periods, characterized by fundamental differences in method, immediate aims, and available mathematical tools.

The first period—the so-called geometrical period—extended from the earliest empirical determinations of the ratio of the circumference of a circle to its diameter to the invention of the calculus about the middle of the seventeenth century. The main effort was directed toward the approximation of this ratio by the calculation of perimeters or areas of regular inscribed and circumscribed polygons.

The second period began in the middle of the seventeenth century and lasted for more than a hundred years. During this period the methods of the calculus were employed in the development of analytical expressions for π in the form of infinite series, products, and continued fractions.

The third period, which extended from the middle of the eighteenth century to nearly the end of the nineteenth century, was devoted to studies of the nature of the number π. J. H. Lambert [2] proved the irrationality of π in 1761, and F. Lindemann [3] first established its transcendence in 1882.

This article is concerned with the second period and its sequel, which extends to the present day.

According to Hobson [1], the first analytical expression discovered in this period is the infinite product

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots,$$

which was published by John Wallis [4] in 1655.

Lord Brouncker, the first president of the Royal Society, about 1658 found the infinite continued fraction

$$\frac{\pi}{4}=\frac{1}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots,$$

which was shown subsequently by Euler to be equivalent to the alternating series

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots,$$

known to G. W. Leibniz in 1674.

The great majority of calculations of π to many decimal places have been based upon the power series

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots,\quad -1\leqq x\leqq 1,$$

which was discovered in 1671 by James Gregory [5]. He failed, however, to note explicitly the special case corresponding to $x=1$, which is ascribed to Leibniz.

* Numbers in brackets refer to the references listed at the end of the article.

Sir Isaac Newton [6] in 1676 discovered the power series

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots,$$
$$-1 \leqq x \leqq 1,$$

which has been used by a few computers of π.

In 1755 Leonhard Euler [7] obtained the following useful series:

$$\arctan x = \frac{x}{1+x^2}\left\{1 + \frac{2}{3}\left(\frac{x^2}{1+x^2}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{x^2}{1+x^2}\right)^2 + \cdots\right\}.$$

It was by means of Gregory's series, taking $x = 1/\sqrt{3}$, that Abraham Sharp [8], at the suggestion of the English astronomer Edmund Halley, computed π to 72 decimal places in 1699, thereby nearly doubling the greatest accuracy (39 decimal places) attained by earlier computers, who had used geometrical methods. Sharp's calculation was extended by Fautet de Lagny [9] in 1719 to 127 decimals (the 113th place has a unit error).

Newton set $x = \frac{1}{2}$ in his series, and thereby computed π to 14 places. A Japanese computer, Matsunaga Ryohitsu [10], used the same procedure to evaluate π correct to 49 decimal places in 1739. About 1800 a Chinese, Chu Hung, calculated π to 40 places (25 correct) by this series [10].

Most computers of π in modern times have used Gregory's series in conjunction with certain arctangent relations. Only nine of these relations have been employed to any extent in such computations. We shall now consider these formulas, arranged according to the increasing precision of the approximations computed by their use.

I. $$\frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}$$

Euler [7] in 1755 used this relation in conjunction with his series for arctan x to compute π correct to 20 decimal places in one hour. Baron Georg von Vega [11] in 1794 employed Gregory's series and the preceding relation to evaluate π to 140 decimal places, of which the first 136 were correct. This precision was exceeded by that attained by an unknown calculator whose manuscript, containing an approximation correct to 152 places, was seen in the Radcliffe Library at Oxford toward the close of the eighteenth century.

II. $$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{70} + \arctan\frac{1}{99}$$

Euler published this relation in 1764. It was used by William Rutherford [12] in 1841 to compute π to 208 places (152 correct).

III. $$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}$$

This formula was supplied the calculating prodigy Zacharias Dahse [13] by L. K. Schulz von Strassnitzky of Vienna. Within a period of two months in 1844, Dahse thereby evaluated π correct to 200 places.

IV. $$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$

First published by Charles Hutton [14] in 1776, this relation was used by W. Lehmann [15] of Potsdam to compute π to 261 decimals in 1853. Tseng Chi-hung [16] in 1877 used the same formula to evaluate π to 100 decimals in a little more than a month.

V. $$\frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7}$$

The relation was also published by Hutton [14] in 1776, and independently by Euler in 1779. Vega [17] used it in 1789 to compute 143 decimals (126 correct). In order to remove the uncertainty caused by the discrepant approximations of Rutherford and Dahse, Thomas Clausen [18] extended the calculation to 248 correct decimals in 1847, and Lehman [15]

reached 261 decimals in 1853 by this formula, confirming his independent calculation of π to the same extent by relation IV. Edgar Frisby [19] in Washington, D. C. used relation V in conjunction with Euler's series to compute π to 30 places in 1872.

$$\text{VI.}\quad \frac{\pi}{4}=3\arctan\frac{1}{4}+\arctan\frac{1}{20}+\arctan\frac{1}{1985}$$

This formula was published by S. L. Loney [20] in 1893, by Carl Störmer [21] in 1896, and was rediscovered by R. W. Morris [22] in 1944. By means of this formula D. F. Ferguson, then of the Royal Naval College, Eaton, Chester, England, performed a longhand calculation of π to 530 decimal places between May 1944 and May 1945. At that time he discovered a discrepancy between his approximation and the final result of William Shanks—discussed under formula IX—beginning with the 528th place. The first notice of an error in Shanks's well-known approximation appeared in a note [22] published by Ferguson in March 1946. He continued his calculation of π and in July 1946 published [23] a correction to Shanks's value through the 620th decimal place. Subsequently, Ferguson used a desk calculator to reach 710 decimals [24] by January 1947, and finally 808 decimals [25] by September 1947.

$$\text{VII.}\quad \frac{\pi}{4}=8\arctan\frac{1}{10}-\arctan\frac{1}{239}-4\arctan\frac{1}{515}$$

S. Klingenstierna discovered this relation in 1730; it was rediscovered more than a century later by Schellbach [26]. It was used by C. C. Camp [27] in 1926 to evaluate $\pi/4$ to 56 places. D. H. Lehmer [28] recommended it in conjunction with the next formula for the calculation of π to many figures. G. E. Felton on March 31, 1957 completed a calculation of π to 10021 places on a Pegasus computer at the Ferranti Computer Centre in London. This required 33 hours of computer time. The result was published to 10000 places [29]. A check calculation using formula VIII revealed that, because of a machine error, this result was incorrect after 7480 decimal places.

Gauss [30] investigated the derivation of arctangent relations and reduced it to a problem in Diophantine analysis. Relation VIII is one of several formulas he developed. J. P. Ballantine [31] substantiated Lehmer's claim that this formula is especially effective for extensive calculation, by discussing its use in conjunction with Euler's series for the arctangent.

$$\text{VIII.}\quad \frac{\pi}{4}=12\arctan\frac{1}{18}+8\arctan\frac{1}{57}-5\arctan\frac{1}{239}$$

Felton carried out a second calculation to 10021 places, and by March 1, 1958 had removed all discrepancies from his results, so that the approximations computed from formulas VII and VIII agreed to within 3 units in the 10021st decimal place. The corrected result remains unpublished.

$$\text{IX.}\quad \frac{\pi}{4}=4\arctan\frac{1}{5}-\arctan\frac{1}{239}$$

This is the most celebrated of all the relations of this kind. John Machin, its discoverer, computed π correct to 100 decimals by means of it in conjunction with Gregory's series, and the result [32] appeared in 1706. Clausen [18] in 1847 used this relation in addition to Hutton's formula V to compute π correct to 248 decimal places, as has already been noted.

Rutherford resumed his calculation of π in 1852, using Machin's formula this time, as did his former pupil William Shanks. Shanks's first published approximation to π contained 530 decimal places, and was incorporated in Rutherford's note [33], published in 1853, which set forth his approximation to 441 decimals.

Later that year Shanks published his book [34] containing an approximation to 607 places and giving all details of the calculation to 530 places. It is now known that Shanks's value was incorrectly calculated beyond 527 decimal places. The accuracy of that value was further vitiated by a blunder committed by Shanks in correcting his copy prior to publication, with the result that similar errors appear in decimal places 460–462 and 513–515. These errors persist in Shanks's first paper of 1873 [35] containing the extension to 707 decimals of his earlier approximation. His second paper of that year [36], which contained his final approximation to π, gives corrections of these errors; however, there appears an inadvertent typographical error in the 326th decimal place of his final value. In retrospect, we now realize that Shanks's first value published in 1853 was the most accurate he ever published.

The accuracy of Shanks's approximation to at least 500 decimals was confirmed by the independent calculations of Professor Richter [37] of Elbing, Germany, who in 1853–1854 computed successive approximations to 330, 400, and 500 places. Richter's communications do not reveal the formula that he used.

Machin's formula was used by H. S. Uhler in an unpublished computation correct to 282 places, which was completed in August 1900.

F. J. Duarte computed π correct to 200 places by this method in 1902. The result was published [38] six years later.

As a by-product of his calculation of the natural logarithms of small primes, Uhler in 1940 noted [39] confirmation to 333 decimal places of Shanks's approximation.

In December 1945, Professor R. C. Archibald suggested that the writer undertake the computation of π by Machin's formula in order to provide an independent check of the accuracy of Ferguson's calculations. With the collaboration of Levi B. Smith, who evaluated arctan 1/239 to 820 decimal places, the writer computed π to 818 places by February 1947, using a desk calculator. The result was published [24] to 808 places in April 1947, and was verified to 710 places by Ferguson in a note published concurrently [24]. The limit of 808 decimals in the published value was chosen to provide precision comparable to that obtained by P. Pedersen [40] in his approximation to e.

Collation of this 808-place approximation with results obtained by Ferguson later that year revealed several erroneous figures beyond the 723rd place in the writer's approximation to arctan $\frac{1}{5}$. These errors vitiated the corresponding figures in the approximation to π. Corrections of these errors and extensions of Ferguson's results appeared in a joint paper [25] by Ferguson and the writer in January 1948, which concluded with an 808-place approximation to π of guaranteed accuracy.

Subsequently, Smith and the writer resumed their calculations and by June 1949 had obtained an approximation to about 1120 decimal places [41]. Before final checking of this extension could be completed, the ENIAC (Electronic Numerical Integrator and Computer) at the Ballistic Research Laboratories, Aberdeen Proving Ground, was employed by George W. Reitwiesner and his associates in September 1949 to evaluate π to about 2037 places (2040 working decimals) in a total time (including card handling) of 70 hours [42]. Machin's formula was also used in this computation.

In November 1954, Smith and the writer extended their calculation to 1150 places, and in January 1956 reverted to this work once more to attain their final result, which was terminated at 1160 places, of which the first 1157 agree with those obtained on the ENIAC.

A calculation of π was performed in duplicate on the NORC (Naval Ordnance Research Calculator) in November 1954 and in January 1955 as a demonstration problem, prior to the delivery of that computer to the U. S. Naval Proving

Grounds at Dahlgren, Virginia. Again, Machin's formula was selected, and the calculation was completed to 3093 decimal places in 13 minutes running time. A report of this work, in which the value of π was presented unrounded to 3089 decimal places, was published by S. C. Nicholson and J. Jeenel [43] of the Watson Scientific Computing Laboratory, in New York.

In January 1958, François Genuys [44] programmed and carried out the evaluation of π correct to 10000 decimal places on an IBM 704 Electronic Data Processing System at the Paris Data Processing Center. Machin's formula in conjunction with Gregory's series was used. Only 40 seconds were required to attain the 707 decimal-place precision reached by Shanks, and one hour and forty minutes was required to reach the 10000 places of the final result.

On July 20, 1959, the program of Genuys was used on an IBM 704 system at the Commissariat à l'Energie Atomique in Paris to compute π to 16167 decimal places. This latest approximation is unpublished at present.

The motivation of modern calculations of π to many decimal places was conjectured by Professor P. S. Jones [45] in 1950 as being attributable to "intellectual curiosity and the challenge of an unchecked and long untouched computation." This reason for undertaking such work should be supplemented by reference to the recurrent interest in determining a statistical measure of the randomness of distribution of the digits in the decimal representation of π.

Augustus De Morgan [46] drew attention to the deficiency in the number of appearances of the digit 7 in Shanks's 607-place approximation to π. In 1897 E. B. Escott [47] raised the question whether the deficiency of 7's noted in Shanks's final approximation could be explained.

In June 1949, the late Professor John von Neumann expressed an interest in utilizing the ENIAC to determine the value of π and e to many places as the basis for a statistical study of the distribution of their decimal digits. A statistical treatment of the first 2000 decimal digits of both π and e was published by N. C. Metropolis, G. Reitwiesner, and J. von Neumann [48]. Further analysis of these data was performed by R. E. Greenwood [49], using the coupon collector's test. A

TABLE 1

CUMULATIVE DISTRIBUTION OF THE FIRST 16000 DECIMAL DIGITS OF π

THOUSAND	DIGIT									
	0	1	2	3	4	5	6	7	8	9
1	93	116	103	102	93	97	94	95	101	106
2	182	212	207	188	195	205	200	197	202	212
3	259	309	303	265	318	315	302	287	310	332
4	362	429	408	368	405	417	398	377	405	431
5	466	532	496	459	508	525	513	488	492	512
6	557	626	594	572	613	622	619	606	582	609
7	657	733	692	686	702	730	708	694	680	718
8	754	833	811	781	809	834	816	786	764	812
9	855	936	911	884	910	933	914	883	854	920
10	968	1026	1021	974	1012	1046	1021	970	948	1014
11	1070	1099	1111	1080	1133	1150	1129	1070	1031	1127
12	1162	1193	1214	1176	1233	1262	1227	1166	1144	1223
13	1266	1314	1316	1272	1343	1358	1324	1260	1246	1301
14	1365	1416	1419	1383	1440	1455	1426	1344	1339	1413
15	1456	1513	1511	1491	1553	1549	1520	1441	1458	1508
16	1556	1601	1593	1602	1670	1659	1615	1548	1546	1610

count of each of the decimal digits appearing in the NORC approximation appears in the paper of Nicholson and Jeenel [43]. A number of recent investigators have discussed the distribution of digits in Shanks's approximation and in the corrected value of π. These investigators include F. Bukovszky [50], W. Hope-Jones [51], E. H. Neville [52], and B. C. Brookes [53].

The writer has recently completed a count by centuries of the 16167 decimal digits constituting the fractional part of the latest approximation to π. An abridgment of this information is presented in the accompanying table.

The standard χ^2 test for goodness of fit reveals no abnormal behavior in the distribution of digits in this sample; in particular, there appears to be no basis for supposing that π is not simply normal [54] in the decimal scale of notation. It has been pointed out recently by Ivan Niven [55] that the normality of such numbers as π, e, and $\sqrt{2}$ has yet to be proved.

Numerical studies directed toward the empirical investigation of the normality of π clearly require increasingly higher decimal approximations, which can best be obtained by use of ultra-high-speed electronic computers now under design and development.

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Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Fundamentals of Mathematics, Elbridge P. Vance (Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1960). Cloth, x+469 pp., $7.50.

The author attempts to give a unified treatment of some of the basic ideas of algebra, trigonometry, and analytic geometry, together with a fairly substantial introduction to the calculus. The work covered in this text should prepare the student fairly well for further study in the area of mathematical logic, finite mathematics, probability and statistics, or the calculus.

The content is apparently designed for a full year of approximately ten semester hours of freshman college mathematics. It is not a radical departure from the traditional textbooks used in first-year college mathematics, yet it does give the student a better perspective of mathematics as a unified discipline. The traditional college algebra, trigonometry, analytic geometry, and the calculus are, of course, not given in their usual breadth and depth of treatment, but the author has quite skillfully included the essentials in these areas. This type of text is especially valuable in those schools in which early introduction to basic concepts is necessary for application in related subjects.

To acquaint the reader with the general outline of the text, the chapter headings are as follows:

1. Numbers and Elementary Operations
2. Fractions, Exponents, and Radicals
3. Coordinate Systems, Functions, and Graphical Representation
4. The Circular Functions (Trigonometry)
5. Linear Functions
6. Quadratic Functions
7. Determinants
8. Mathematical Induction and the Binomial Theorem
9. Functions, Limits, and Continuity
10. Derivatives
11. Polynomial Functions
12. Some Applications of Differentiates
13. Inverse Functions
14. Exponential and Logarithmic Equations
15. Application of the Circular Functions in Solving Triangles
16. Further Application of Circular Functions
17. Complex Numbers
18. Conic Sections
19. Integrations
20. Permutations, Combinations, and Probability

In the opinion of the reviewer, this is more material than can be covered thoroughly in the time usually devoted to a first-year college mathematics course.

There are several unusual features of the text. The brief discussion of the one-dimensional coordinate system leads very nicely to the ordinary rectangular coordinate system. The definitions of the circular functions in terms of this system is a basic unifying link between trigonometry and analytic geometry. The reviewer is glad to see that the analytic, rather than the computational, aspects of trigonometry are given emphasis. Although the work in mathematical induction is treated in an excellent manner, the reviewer objects to wide-sweeping generalizations, as: "No certain conclusions can be reached by the inductive logical process." Such a statement does not make allowance for a structure in which all possible cases under consideration can be displayed.

In general, the definitions are quite carefully expressed. In some instances there is a lack of rigor. This is evidenced by a statement such as: "A positive integer is called *composite* if it can be expressed as the product of two or more positive integers, which are its factors." Here two would be composite since it can be expressed as the product of one and two. Generally, however, there is judicious use of rigor for the level for which the text is intended.

For the most part, the text is a good one, and the reviewer does recommend it for serious consideration for a freshman college mathematics course when a unified year of mathematics is desirable.—*T. E. Rine, Illinois State Normal University, Normal, Illinois.*

Introduction to Higher Mathematics for the General Reader, Constance Reid (New York: Thomas Y. Crowell Company, 1960). Cloth, v+184 pp., $3.50.

This book does just what the title suggests. It *introduces* the general reader to the topics included in higher mathematics. It tells just enough to whet the interest of the reader.

In many book titles, the word "introduction" indicates a text for the first course of a subject. This book is not a text in any sense, nor is it so intended. It is a brief (184 pp.) and interesting survey, giving an historical background and a simple, lucid development of the various topics. As such, it would make good supplementary reading—for the high school teacher interested in gaining a better understanding of these topics, for advanced high school students interested in exploring what lies ahead for them in the field of mathematics, and for the "general reader" who is not a professional mathematician but who enjoys mathematics.

The book starts with a look at some simple numbers: natural numbers, zero, integers, ra-

tional numbers, etc. The reader is shown the need for the development of imaginary numbers in order to solve a simple equation such as $x^2+1=0$. It is further shown, however, that we need develop no new kinds of numbers beyond complex numbers for solving any algebraic equation, even one such as $x^3-i=0$.

The Laws of Arithmetic are examined as the springboard to observing how certain operations vary from these laws; e.g., "multiplication" of matrices fails to follow the Commutative Law. From this the reader is led to an examination of groups, with good illustrations being given of both commutative and noncommutative groups.

Starting with Chapter 6, the development and relationship of the various geometries are described. Euclidean geometry is shown as the geometry of flat surfaces, but of special value for giving us the axiomatic method. The non-Euclidean geometries are the geometries of curved surfaces, but of particular importance for having broken the bonds of tradition. As the author states, "The invention of non-Euclidean geometry had very much the same effect on mathematics as the invention of non-commutative algebra, which it actually preceded by a few years. Both freed mathematics from the tyranny of the 'obvious,' the 'self-evident,' and the 'true'. . . . " Analytic geometry, with its relating of geometric figures to algebraic expressions, is used as an avenue for discussing fourth and higher dimensions. Finally, the reader is introduced to topology with its emphasis on the spatial relationship between figures.

The last four chapters consist of a brief discussion of the calculus, a fairly difficult chapter on point set theory, a well-written introduction to the subject of symbolic logic, and the concluding chapter entitled "Mathematics, the Inexhaustible." The "happy ending" to this little volume is the comforting thought that it has been proved that no "logic machine" can be built which can solve all mathematical problems, and hence there will continue to be a need for mathematicians.—*Donald C. Martinson Wichita High School North, Wichita, Kansas.*

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

NINETEENTH CHRISTMAS MEETING

December 27–30, 1960
Arizona State University, Tempe, Arizona
Lehi T. Smith, Arizona State University, Tempe, Arizona

THIRTY-NINTH ANNUAL MEETING

April 5–8, 1961
Conrad Hilton Hotel, Chicago, Illinois
Hobert Sisler, Morton High School West, 2400 Home Avenue, Berwyn, Illinois

JOINT MEETING WITH NEA

June 28, 1961
Atlantic City, New Jersey
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTY-FIRST SUMMER MEETING

August 21–23, 1961
University of Toronto, Canada
Father John C. Egsgard, C.S.B., St. Michael's College School, 1515 Bathurst Street, Toronto 10, Ontario, Canada

Other professional dates

Men's Mathematics Club of Chicago and Metropolitan Area

December 16, 1960
January 20, 1961
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

The Greater Cleveland Council of Teachers of Mathematics

January 19, 1961
Western Reserve University
Bessie Kisner, Strongsville High School, Strongsville, Ohio

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Author index

Title index

Subject index